Third Misconceptions Seminar Proceedings (1993)

Paper Title: Student Intuitions of Lines: Exploring their origins, uses, and interactions
Author: Chiu, Ming M.

Abstract: In this study, I examined intuitive conceptions in geometry, focusing on their origins, uses, and interactions. Data included audio taped interviews of sixteen middle school students during pre- and post-tests. When asked to rank several paths between two points according to length, these students invoked four intuitive criteria: compression, detour, complexity, and straightness. My analysis of their explanations suggests that these intuitive conceptions originated from everyday experiences (such as motion).

The students productively applied these intuitions to simple comparisons of paths (e.g. straight line vs. staircase), but spontaneously recognized their inadequacy for more difficult comparisons. Then, I taught them a new strategy: rearranging the linear pieces of the paths into horizontal and vertical components. In their post-test with additional paths three weeks later, most students continued to use their intuitions. After recognizing their inadequacy again, they independently and successfully applied their new strategy.

In both pre- and post-tests, many students invoked multiple intuitions when comparing two paths. They tried to resolve these intuitions' interactions by ranking them and by integrating them.

Keywords: Concept Formation,Philosophy,,Misconceptions,Cognitive Structure,Constructivism,,,
General School Subject: Mathematics
Specific School Subject: Geometry
Students: Junior High School

Macintosh File Name: Chiu - Lines
Release Date: 10-16-93 A, 11-4-1994 I

Publisher: Misconceptions Trust
Publisher Location: Ithaca, NY
Volume Name: The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics
Publication Year: 1993
Conference Date: August 1-4, 1993
Contact Information (correct as of 12-23-2010):
Web: www.mlrg.org
Email: info@mlrg.org

A Correct Reference Format: Author, Paper Title in The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics, Misconceptions Trust: Ithaca, NY (1993).

Student Intuitions of Lines:
Exploring their origins, uses, and interactions

Ming M. Chiu
Education, Math, Science, and Technology
University of California, Berkeley
USA

ABSTRACT

In this study, I examined intuitive conceptions in geometry, focusing on their origins, uses, and interactions. Data included audio taped interviews of sixteen middle school students during pre- and post-tests. When asked to rank several paths between two points according to length, these students invoked four intuitive criteria: compression, detour, complexity, and straightness. My analysis of their explanations suggests that these intuitive conceptions originated from everyday experiences (such as motion).

The students productively applied these intuitions to simple comparisons of paths (e.g. straight line vs. staircase), but spontaneously recognized their inadequacy for more difficult comparisons. Then, I taught them a new strategy: rearranging the linear pieces of the paths into horizontal and vertical components. In their post-test with additional paths three weeks later, most students continued to use their intuitions. After recognizing their inadequacy again, they independently and successfully applied their new strategy.

In both pre- and post-tests, many students invoked multiple intuitions when comparing two paths. They tried to resolve these intuitions' interactions by ranking them and by integrating them.

1.0 INTRODUCTION

A programmer manipulates a computer's memory like a blackboard, writing and erasing as needed. But can a teacher simply "insert" or "delete" information from students' minds? Researchers in the fields of education, psychology, and linguistics argue persuasively that a teacher can not because students invoke intuitive knowledge that can either facilitate or hinder their learning. Since the existence of student intuitions is gaining recognition, researchers and educators must push further to explore their origins, uses, interactions and ultimately specify their role in learning and development.[1]

This paper begins by reviewing some of the research on intuitions. Then, I present a study of sixteen middle school students using their intuitions to compare the relative lengths of two dimensional paths. After discussing specific

[1]Intuition and intuitive concept are used synonymously.

intuitive concepts and their applications, I examine the students' uses of multiple intuitions in different situations. Finally, I conclude with some implications for instruction.

2.0 THEORETICAL PERSPECTIVE

2.1 WHAT IS AN INTUITION?

Researchers from various fields have argued that students have intuitive concepts that typically 1) originate prior to schooling, 2) conflict with expert ideas, and 3) resist attempts to circumvent them. Piaget and his colleagues (Piaget & Inhelder, 1948/1967; Piaget, Inhelder & Szeminska, 1948/1960; Piaget & Szeminska, 1941/1952) showed that children responded to questions about space, geometry, and number with consistent answers that varied significantly from those of adults. Research in the areas of psychology (Carey, 1985, 1988; Ogborn & Bliss), linguistics (Talmy, 1988; Lakoff, 1992), history of science (Kuhn, 1965), and philosophy (Johnson, 1987) have further buttressed this claim. In mathematics education in particular, Davis and Vinner (1986), Nesher (1987), Mason (1989), Fischbein, Deri, Nello, & Marino (1985), Shaughnessy (1977), and Clement (1982) have demonstrated that students' intuitions persisted despite instruction in calculus, subtraction, polygons, multiplication, probability, and algebra. Numerous studies documenting the prevalence of intuitions in diverse disciplines (Confrey, 1990; Eylon & Linn, 1988) have highlighted the importance of understanding this phenomena in order to develop appropriate instructional strategies.

I further explicate the term "intuition" by contrasting it with a prototypical "formal concept" in six ways: origin, societal support, explication, systematicity, and justification (see Table 1). Consider the formula "the area of a triangle = 0.5 x base x height" and the intuition "taller things are bigger." In contrast to the formal teaching of the expert-generated triangle formula in school, students discover the height-volume relation from their own experiences without institutional instruction (Confrey, 1990). Furthermore, the formula is clearly delineated, denoted and defined in relation to other mathematical concepts as part of a coherent and consistent system (Vygotsky, 1934/1986). On the other hand, the intuition lacks precise articulation and may not be strongly tied to other notions (formal or intuitive). Finally, teachers justify the validity of the triangle formula by appealing to its systematic coherence with other mathematical concepts and to their authority as representatives of both the society and the discipline. In contrast, when students justify the taller-bigger intuition, they point to the purposeful reality of their physical and social experiences. **In short, an intuition is a relatively unarticulated and sparsely connected notion arising from and justified by a person's experience without the aid of formal instruction.**

Table I. Comparison of formal concepts and intuitions

	Formal concept	Intuition
Origin	Expert-generated	Self-discovery
Societal Support	Teacher	Little or none
Explication	Precise	Relatively unarticulated
Systematicity	Tightly connected	Sparse links
Justification	Systematicity + Authority	Personal experience

2.2 INVOKING INTUITIONS IN UNFAMILIAR SITUATIONS

People facing a problem in an unfamiliar domain may use their intuitions rather than general purpose strategies (such as means-ends analysis (Newell & Simon, 1972)). The prevalence of intuitions in daily life endows them with a credibility that encourages their use outside of their usual contexts. In an unfamiliar situation, a person without a clear method of tackling a problem may turn to any intuition that has seems promising to him or her. As a result, the person's intuitive solution attempt may prove ineffective. Since an intuition may accumulate credibility through successful uses in everyday activities, the person continues to perceive it as successful. On the other hand when a person applies an intuitive concept successfully, he or she may not systematically apply it to isomorphic problems because s/he may not recognize the critical components in his/her problem solution. When a person uses an intuition in unfamiliar situations, failures can be negated by repeated successes in everyday activities, and successes are not necessarily replicable.

2.3 INTERACTIONS BETWEEN INTUITIONS

Many studies have focused on specific intuitions, but few have examined how multiple intuitions interact when invoked within a particular situation. Hewson (1981) and diSessa (1983; 1988; in press) argue that students may resolve conflicts between intuitions by raising or lowering the status of one relative to the other. In particular, diSessa describes a knowledge network with two types of priorities, cueing and reliability, for each student's intuition. The cueing priority quantifies the degree of "fit" between the situation's salient features and the intuitive concept. After an intuition is cued, the reliability priority helps determine its applicability by providing feedback based on other knowledge, such as mathematical principles and other intuitions.

Consider the Piagetian problem of pouring liquid from a short, wide beaker into a tall, thin cylinder. Assumr that a child also knows that "wider things are bigger." The child then cues both intuitive concepts due to the beaker and the cylinder's salient size differences. The "taller" intuition may have a higher reliability priority because most of the big things that he/she's seen are tall. The child then promotes the status of the "taller" intuition and demotes that of the "wider" intuition, thereby concluding that there's more liquid in the taller cylinder (in the absence of other competing views such as a conservation of matter principle). On the other hand, the child may compare a tall, wide beaker and a short, narrow cup, both filled with liquid. In this situation, the same intuitions can promote each other's status and support the conclusion that the beaker contains more liquid. Feedback about the conclusion then serves to modify the cueing and reliability priorities appropriately. So multiple intuitions may interact as a person compares their fit to the current situation and the history of their successes and failures.

2.4 COHERENCE OF INTUITIONS

The research perspectives on the coherence (the intricacy and density of connections) among intuitions span a continuum from the virtual vacuum (Skinner) to the tightly-woven theory (Carey).

Skinner (1954) argues that learning was nothing more than chains of behavioral responses to stimuli. Given the appropriate reinforcement and "shaping" by an instructor (or a machine), a student could learn any behavior. His teaching program ignores prior knowledge because each chain of behavioral responses is essentially separate from each other unless integrated by appropriate stimuli. Hence, early learned behaviors are largely unrelated, if not isolated.

diSessa (1983, 1988, in press) argues that students generalize from common experiences to create loosely-connected, heterarchical pieces of information. Connections are formed from the serendipitous juxtapositions of these experiences mediated by concerns such as agency and causality. As students progress toward expertise, they build additional connections to systematically structure their knowledge (diSessa, 1983, 1988, in press; Hawkins, Apelman, Colton & Flexner, 1982). Although novices may respond to unfamiliar questions inconsistently, diSessa (1983, 1988, in press) argues that they use only a limited set of intuitions in contrast to the infinite range of guesses in Skinner's view.

Talmy (1988) proposes a systematic, dimensional ordering of intuitions that involve forces. Each intuition sits at an intersection of an n-dimensional matrix with parameters such as relative strength and result. Although Talmy does not present a developmental mechanism for these connections, one might speculate that additional knowledge builds upon a small core of dimensions and intuitions. Extrapolating along this view of intuition, novices may answer incorrectly initially, but will move toward a predictable final result along its known dimensions, unlike the unpredictability of Skinner and diSessa's views.

Finally, several researchers argue that students construct systematic, intuitive theories. Many physics researchers (McCloskey, 1983; Halloun & Hestenes, 1985; Nersessian & Resnick, 1984) point to a variant of "impetus" theory as the intuitive theory for physical motion. Although the proponents of this view generally limit their scope to physics, Carey (1985, p. 200) has conjectured that infants "are innately endowed with two theoretical systems: a naive physics and a naive psychology." In this view, student's intuitions are integrated into a coherent system in which reasoning flows smoothly rather than jumping from one knowledge piece to another. Consequently, students should answer questions predictably and consistently (but possibly incorrectly).

In short, the coherence of one's intuitions may be practically non-existent, minimally clustered, dimensionally structured, or tightly woven.

2.5 RESEARCH QUESTIONS FOR EXAMINING GEOMETRY INTUITIONS

Thus far, I have speculated on the source of intuitions, the nature of their uses, and some possible interactions. The remainder of this paper addresses these issues through my analysis of student interviews. In particular, what intuitions do these students invoke to compare lengths of two dimensional lines? Where do they originate? How do they use these intuitions? Do these students invoke their intuitive concepts consistently across different situations? Do they employ multiple intuitions? If so, how do they interact? What happens to their intuitions when students learn a formal strategy?

3. 0 METHOD

In this study, I analyzed two problem solving sessions, possibly with instruction at the end of the first one. Three weeks separated the first and second problem sessions. The format of the individual interviews drew upon Piaget's revised clinical method (Piaget & Szeminska 1941/1952). Data collection included audio tape, field notes and students' written work.

The analysis proceeded as follows. After transcribing the audio tapes, I incorporated recorded gestures from my field notes into the dialogue (so the temporal placement of the gestures is only approximate). The coding consisted of two parts. The open coding (Strauss, 1987) identified different intuitions and their uses, forming tentative categories. Through axial coding (Strauss, 1987), I examined variations among the intuitions employed and refined the categories by looking for variations along particular dimensions (See Appendix A). Finally, a second coder used my criteria to recode the entire transcript.

3.1 STUDENTS AND SETTING

The sixteen students (twelve- to fourteen-year-olds) attended a summer remedial program for Asian-American middle school students. This program draws its target population from seven different local public middle schools. Although fifteen of the students immigrated to the United States between three to five years ago, they all spoke English fluently. All the students were taking first year algebra in the fall, and none had taken geometry yet.

I assisted the teacher during sports activities and knew several of the students from the previous year. During study period, I interviewed individual students in a separate classroom without any time constraints.

3.2 PROCEDURE

Three transparencies sitting separately on a table greeted each student (see fig. 1). The interviewer then demonstrated that each path connects two points on a fourth transparency. Next, he asked them to "compare their lengths."[2]

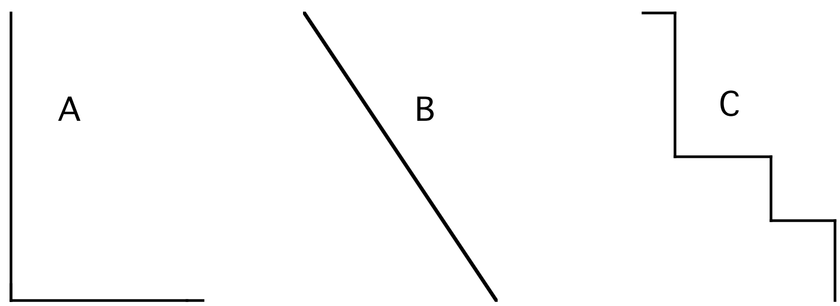

Figure 1. L (A) , diagonal (B), and staircase (C) paths

Most middle school students knew that the diagonal line is the shortest, but many had difficulty determining that the staircase and L paths have the same length. If a student did not solve the problem, the interviewer provided a ruler, graph paper, index cards, paper clips, rubber bands, string, scissors, and tacks to help him/her. If a student did not present a solution that included aligning and comparing the equivalent lengths of the non-linear paths' horizontal and vertical components, the interviewer provided "scaffolding" (by adapting the explicitness of the question to the student's progress, (Rogoff & Gardner, 1984; Vygotsky, 1978; Wood, Bruner & Ross, 1976; Wood, Wood & Middleton, 1978) to encourage the student to construct the "align and compare" solution. (See Fig. 2, Appendix B).

[2]Although I refer to the lines as paths in this paper, I only used their letter names when discussing them with the students.

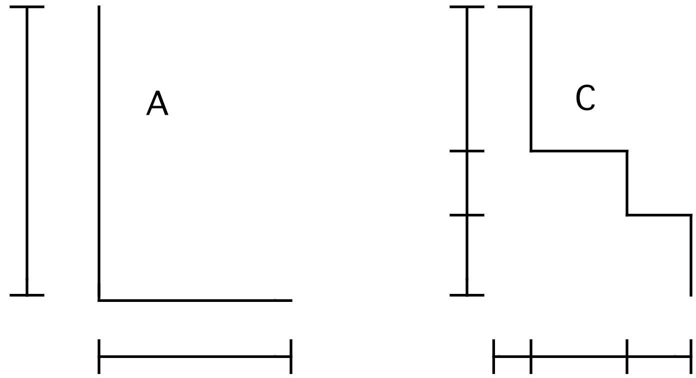

Figure 2. Align and compare solution for L and staircase paths

Three weeks later, each student received a similar question, also on transparencies (see fig. 2). This problem included two additional paths: a zigzag path (B) and a staircase with an incline section (D). Moreover, in this problem, the transparencies were initially on top of one another, not separated as in the first session.

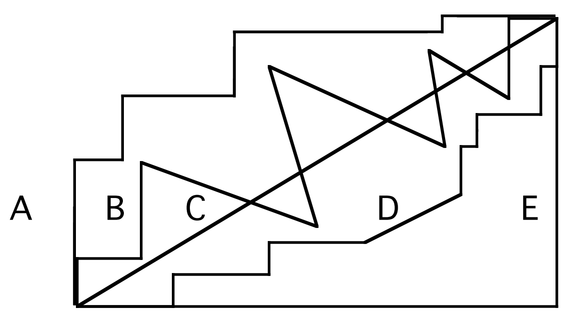

Figure 3. Staircase (A) , zigzag (B) , diagonal (C), staircase with incline (D) , and L paths (E)

To solve this problem, the students could align each segment in the zigzag path to a part of either a horizontal or a vertical segment in the L-path and compare the differences. For instance, each of the zigzag's segments was either equal to greater than its L-path counterpart. overlapping one another when aligned with corresponding horizontal and vertical segments of the L-path (see fig 4).[3]

[3]Each zigzag segment must be aligned with either a horizontal or vertical part of the L-path and all parts of the L-path must be included in the comparison.

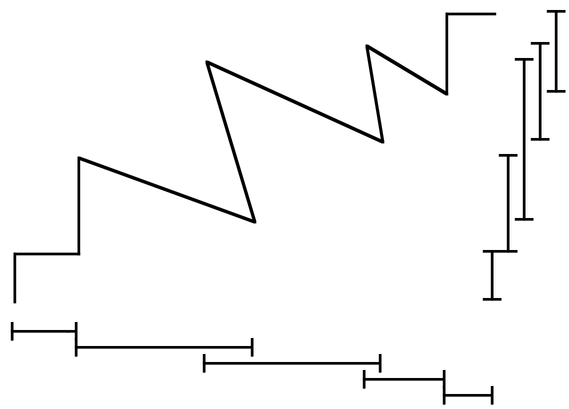

Figure 4. Align & compare solution for the zigzag path

Similarly, the staircase with incline path was shorter than the L-path (see fig. 5). Except for the incline and the L-path's corresponding horizontal and vertical segments, all the other segments can be aligned as before to demonstrate that they have equal corresponding lengths. The remaining segment, the incline, forms a triangle with the two remaining segments of the L-path. Since the incline is the hypotenuse, it must be shorter than the sum of two L-path lengths. Consequently, the staircase with incline path is shorter than the L-path.

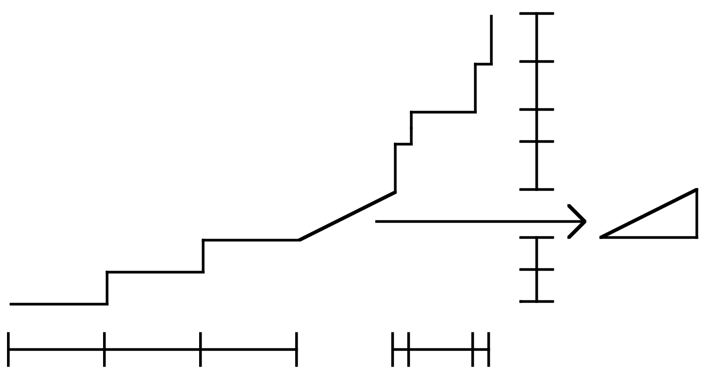

Figure 5. Align & compare solution for
the staircase with incline path.

4. 0 RESULTS AND DISCUSSION

I begin by discussing four central intuitions that students used to solve the problem. Next, I discuss their uses in different situations. Finally, I examine the interactions of multiple intuitions.

4.1 FOUR INTUITIONS

In coding these interviews, I categorized students' comments as intuitions if they 1) justified a answer, 2) stemmed from their own experience, 3) did not appeal to authority, and 4) did not appear in standard mathematical textbooks. Measuring with a ruler or using the "align & compare" method would not be intuitions, for example.

4.1.1 COMPRESSION

Several students argued that some of the paths had been compressed and thus were longer than they appeared. Many familiar every day objects are compressed, so that one can increase their size with little effort. For instance, curly hair, telephone cords, balls of strings, and folded napkins are only a few examples of compressed objects. By pulling the ends of a telephone cord apart, one can apparently increase its length. HA, like most students who invoked this concept, stretched out compressed paths to reveal their actual length.

[HA, I, 119-123][4]

HA: I think this [staircase, C] is longer [than the straight path, B], um, this [straight path, B] is like a piece of string, right?
I: uh-huh.
HA: And this [staircase] is like a piece of string, too except this [staircase, C] is shaped like this [staircase, C], so if you pulled it out, then, it would be longer than B [straight].

HA imagined the static line to be a piece of string which could be manipulated. By pulling out the staircase path, he claimed that it was longer than the straight path.

4.1.2 DETOUR

Several students viewed the lines on the transparencies as motions.[5] Through crawling, walking and running, children learn that moving away from their destination increases the distance (and time) that they travel. To compare two lines, some students imagined objects moving along them and argued that

[4]Student, session number, line number]
[5]Lakoff (19??) argues that this is an instance of a standard image schema transformation from a one dimensional path to a trajector moving along that path.

paths with greater detours from the destination were longer because they required more space to reach their goal.

[FF, II, 193-196]
I: How about E?
FF: B and E? Oh, great, they could be the same, too. Wait, B's longer cause it slants more, you know, like it goes away more, so B's longer.

FF explained that the zigzag path (B) was longer than the L-path (E) because the zigzag path "slants more" and "goes away more" from its destination. Therefore paths that detour further from the destination are longer.

4.1.3 COMPLEXITY

Several students used a complexity criteria to argue that paths composed of more segments are longer. Often, one can compare the sizes of two wholes by counting their constituent parts. Waiting in line behind more people generally requires more time. Houses with more rooms tend to be larger. A post office seven blocks away is probably farther than one that is three blocks away. Even though these units --waiting time per person, rooms, and blocks -- do not describe uniform quantities, counting them often yields sufficient approximations for comparisons.

[LI, I, 1-14]
I: Compare the lengths of A [L path] , B [diagonal path], and C [staircase path] .
LI: What do you mean?
I: Longer, shorter.
LI: Oh, which one is longer?
I: Let's take two of them at a time.
LI: I think this [diagonal path] is the shortest, this [staircase path] the longest, and this [L path] is in-between.
I: Why is that?
LI: Because this one [diagonal path] looks like it's about this size [vertical segment of L-path A] except there's like this [horizontal segment of L-path A] right here. So it would go down [rotates horizontal segment of A [L-path, so that it is collinear with the vertical segment of A], so this [L path] would be longer than this [diagonal path]. And this [staircase path] I don't know but it just seems to have a lot of lines, so it seems more longer.

LI immediately ordered the paths, and confidently demonstrated that the L-path was longer than the diagonal. L may have viewed the staircase path as a gestalt, saying that the staircase path "just seems to have a lot of lines, so it seems ... longer." The staircase seems to be the longest because it is the most complex.

Many students counted the total number of lines after their complexity explanation, so they did not know the exact number of subunits (usually line segments). Instead, they could have relied on a general gestalt to determine the relative complexity of one path to another. [6] Complexity, then, precedes number.

4.1.4 STRAIGHTNESS

Several students simply stated that the straight line was the shortest without further justification. Consequently, I have labeled it as an intuition even though they could have derived it from the other three intuitions. The multiple definitions associated with the word "straight" may also reinforce this concept. In addition to the physical experiences described in the compression and detour sections above (4.1.1 & 4.1.2), the use of "straight there" to minimize distance and both "straight off" and "straight-away" to minimize time support the concept "the shortest path between two points is a straight line." A typical invocation of the straightness intuition follows after FF claims that B [the straight path] is the shortest.

[FF, I, 4-11]

I: Why is B the shortest?
FF: Well, because it's like one straight line. It connects the dots. It's not curved or anything.
I: So why is a straight line better than a curved line --shorter than a curved line?
FF: Well, I mean it takes, I mean, (2)[7] I don't know. (4) I guess because this is smaller, I don't know. The other one [stair case or L-path or curved line?] like takes up more space I guess, is longer.

FF immediately picked out the straight diagonal path as the shortest. She said that the diagonal path was "like one straight line," and contrasted it with lines or objects that are "curved." When asked for further justification, FF hesitated and did not elaborate. Her two "I don't know" statements further suggest that she

---

[6] In the first problem, the relative complexity of the paths can be easily determined because the paths have one, two, and six line segments. In the second problem, the longer paths had 8, 10, and 13 line segments.
[7] numbers inside parentheses [pause] indicates duration of silence in seconds.

did not have a deeper layer of reasoning beyond the recognition of the path's straightness.

In short, students invoked four intuitions to explain their length orderings of paths: compression, detour, straightness, and complexity.

4.2 USING INTUITIONS

This section begins with a summary of the students' intuition use before examining individual student applications in detail.

4.2.1 OVERVIEW OF INTUITION USE

Every student used at least one intuition to solve the problem in the first session (see Table 1). Moreover, 93% of the students began solving the problem with an intuition. However, their intuitions could not help them determine that the L and staircase paths had the same length. Each student also tried other strategies such as visual estimation and correspondence (such as align & compare), but only six students (31%) solved the problem (with "align & compare").

Table II. Percentage of students (N=16) who invoked each intuition at different times

	Session I				Session II			
	1st	2nd	3rd	Total	1st	2nd	3rd	Total
Compression	31%	25%	0%	81%	6%	25%	6%	50%
Detour	6%	19%	6%	31%	13%	0%	19%	31%
Straightness	44%	19%	6%	81%	56%	13%	0%	69%
Complexity	13%	0%	25%	50%	13%	13%	25%	56%
Total	93%	63%	38%	100%	88%	50%	50%	93%
Other	6%	31%	31%		13%	44%	31%	
Average number of intuitions:				2.43				2.06

Note that the total percentages do not total to 100 because some students used only one or two criteria. Likewise, the total percentages for each intuition also includes its use as 4th or later choices.

Despite the inadequacy of their intuitions in the previous session, 88% of these students began the second problem with one of these four intuitions. However, most students (81%) began with a different intuition in the second session. Although they had some limited success, they recognized their intuitions' limitations. Eventually, 83% of the students applied the "align & compare" method successfully not only to the paths they had seen before, but also to at least one of the new paths as well.

No student during either session applied a single intuition systematically across all path comparisons. Only 6 (37%) students applied an intuition to more than one problem in session I, and only two (13%) in session II.

The students' answers reflected not random guesses but recognizable intuitions. Despite the students' use of a successful competitor (align & compare), their inadequate intuitions not only co-existed but continued to demand a higher priority.

4.2.2 INDIVIDUAL USES OF INTUITIONS ACROSS PATH COMPARISONS

None of the students in this study systematically compared all the paths using one intuition in either problem solving session. As discussed earlier, novices do not necessarily recognize the critical aspects in an unfamiliar problem and thus may not apply the same intuition to each problem. Most students bounced from one explanation to the next as they compared pairs of paths without explaining the inadequacy of prior strategies before using new ones. OZ, for example provides four explanations using four different criteria, but each can only be applied to particular path comparisons. OZ begins with a reference to the L-path's longer segment, but continues with a detour explanation.

[OZ, I, 6-12]

I: What do you think of A [L-path] and B [diagonal]? Which is longer?
OZ: A [L-path] is longer than B [diagonal].
I: Why is that?
OZ: Um, one [L-path] is like, it has like a longer line and it goes far out and B [diagonal] is just goes across and is shorter.

OZ then develops an "align & compare" strategy.

[OZ, I, 13-19]

OZ: ... this one is the same.
I: Which two?
OZ: A [L-path] and C [staircase]. Because this line, the one that goes up and down [A's vertical segment], you have this one and this and this one [staircase's, C's three vertical segments] looks the same. And then for the line on the bottom [the L-path A's horizontal segment], you have um, C [staircase] and it goes like that [traces C's horizontal segments], but it's this [the L-path A's horizontal segment] and it's [the L-path, A and staircase, C] the same. It's the same.

However, OZ returned to a compression explanation for his comparison of the staircase and the diagonal paths.

[OZ, I, 20-29]

I: How about C [staircase] and B [diagonal]?
OZ: This is longer. C is longer.
I: Why is C longer?
OZ: Because (2) I think this [staircase] is longer, because if I make this C [staircase] straight, make it straight, it's going to be longer.
I: What you do you mean by making it straight?

OZ: If I like make these [corners of C] straight, then it's gonna be longer.
I: How are you going to make these [corners] straight?
OZ: Like fold it out straight like.

Then, the interviewer asked for additional perspectives on the length comparisons between the three paths. OZ said that he could not apply any of his solutions to another path comparison. Despite his use of several strategies, he could not apply any of them to different path comparisons. Their use depended on the particular situation. Instead of viewing the path comparisons through a single coherent lens, OZ applied different pieces to different situations.[8]

These excerpts also demonstrate the inadequacy of labeling an intuition as correct or incorrect. Students successfully used them in many instances, but not others (e.g. LI in section 4.1.3). By judging them as more or less appropriate to particular situations, we can move beyond right or wrong to more sophisticated criteria such as range of appropriateness, ease of use, and coherence with other ideas.

4.2.3 SUMMARY OF INTUITION USE

These students initially turned to their intuitions to solve the path problem. However, they did not systematically apply them to every path comparison. In addition to bouncing from one intuition to another to compare paths, these students demonstrated the fragmented nature of their intuition use through their extremely limited application range. Moreover, students produced both correct and incorrect results by using their intuitive concepts, so intuitions can not be assessed as simply positive or negative. Despite their intuitions' inadequacies, these students continued using them at the beginning of the second session. Thus, their intuitions had a higher priority than their co-existing successful competitor, the align & compare method that they would later use to solve the problem.

4.3 INTERACTIONS BETWEEN INTUITIONS

[8]His behavior also suggests that he has a particular epistemology that does not seek out or perhaps does not value simple generalizations that apply across many situations. Consequently, he is not making connections between his strategies or between his strategies and these path comparisions.

Most of the students compared paths using one intuition at a time, but a few students (19%) applied two criteria to the same comparison. In this section, I examine how students resolved conflicts and utilized mutually supportive intuitions.

4.3.1 CONFLICTING INTUITIONS

These students typically resolved their conflicts between multiple intuitions by choosing one of them. In the following example, FF's new criteria overrules both of her two previous criteria.

[FF, II, 160-179]

FF: Okay, I guess B [zigzag] is longest of all of them. Yeah, these two are the hardest to compare, B [zigzag] and D [staircase with incline] I mean. Actually, now that I think about it, I think that D [staircase with incline] is longer.

I: Why is D longer?

FF: Because they're more, well you know these [the zigzag's line segments] are slanted, there are only three of them, three slants, and there are like more well,

I: More what?

FF: More like um, like little steps, you know, sort of like stairs, you know more steps, than that one [zigzag].

I: What is it about the stairs?

FF: What do you mean by the stairs? I mean they look like little steps you know. And there are more of them, than this [points to a slant of B]. I mean even though these are slanted. These two [zigzag, B and staircase with incline, D] look like the same. (2) If you straighten this [staircase with incline, D] out, this [staircase with incline, D] will be longer than this line [zigzag, B], so it's [staircase with incline, D's] probably more.

FF believed that the staircase with incline path (D) might be longer than the zigzag path (B) because it was more complex with "more steps." On the other hand, the zigzag path had segments that were "slanted" and that detoured further from the destination than any of the staircase with incline's segments. She did not reconcile the two conflicting conclusions, saying that the zigzag (B) and staircase with incline (D) paths "look the same." By applying a third criteria, compression, she decided that the staircase with incline would be longer than the zigzag path, without reconsidering its relationship to the other criteria.

The overruled detour intuition reappears in a later comparison, though, when I ask her about another path.

[FF, II, 193-196]

I: How about E [L-path]?
FF: B [zigzag] and E [L-path]? Oh, great, they could be the same, too. Wait, E's [L-path] longer cause it, like it goes away more, you know, so E's [L-path] longer.

Even though she had decided against the result of her detour criteria, she uses it again. Intuitions may be virtually invulnerable in this respect because the contradictions occur indirectly at the application level, not at the conceptual level. Two intuitive concepts can generate contradictory results as above. Since the intuitions do not belong to a tightly connected and coherent system however, the student need not re-examine the concepts. As a result, the person can decide that one or both conceptions do not apply to this situation and then, compartmentalize them into different application contexts (Vinner, 1990).

Some students also employed criteria that were less dependent on the particular situation. In this excerpt, FF recognized a violation of the transitive property of equivalence classes.

[FF , II, 144 - 156]

FF: ... So I guess D [staircase with incline] and E [L-path] are the same. [Writes "D same as E"] This one [staircase with incline, D] probably first, well, since they're the same, they're both first, um, the longest. And then, actually these three are the longest [zigzag, staircase with incline, and either staircase or L?] [looks at her written notes] --Wait a minute! It can't be, not right. I messed up, because look, B [zigzag] is equal to --E [L-path] is equal to B [zigzag] right?
I: Hmm.
FF: And then, E [L-path] is equal to B [zigzag]. Wait ... Yeah! A [staircase] is the same as E [L-path], so how can B [zigzag] be longer than A [staircase]? so this [zigzag, B] is the same.

After FF had decided that the staircase with incline path (D) and the L-path (E) were the same, she wrote that down and reviewed her results. She recognized that if E=B and A=E, B>A is not possible. However, she did not backtrack to determine the source of her error. Instead, she decided that B must be the same length as A perhaps because the relationship between A and B must conform to the other salient equalities. Thus, her concern for consistency provided a criterion outside this particular situation. However, as indicated in the prior excerpt [FF, II, 193-196], she switched her answer later. In the course of the session, she wavers back and forth and finally concludes that everything is the same length based on her transitive equalities from above

[FF, II, 291-292]

FF: I guess they're the same. Oh, I don't know. I guess they're the same, because E=B and A=E and D=B, and C is the shortest.

This wavering reinforces a weak coherence view of intuitions because a coherent system would eventually determine a course of action toward a single final result. As a result, this data supports Hewson (1981) and diSessa's (1983,1988) view that intuitions resemble pieces of knowledge.

Students typically resolved conflicts between intuitions by choosing one over the other or by invoking constraints involving prior results. However, students continued to use failed intuitions in later comparisons and "waffled" between contradictory conclusions, demonstrating the fragmented nature of their intuitions.

4.3.2 INFERENCES SUPPORTED BY TWO INTUITIONS

Reasoning along different intuitions may converge towards one conclusion, as FF demonstrated at the beginning of her second session.

[FF, II, 9-19]

I: Which ones are you looking at?
FF: I'm looking at A [staircase]. And then well, I guess either B [zigzag] or D [staircase with incline] is the longest.
I: Why do you think that?
FF: Because there are more zigzags in them, like instead of just going straight across [and up], it [zigzag] slants down, so it's longer.
I: What do you mean by zigzags?
FF: Like you know, like this [points to a corner of the zigzag path]
I: Okay.

Initially, FF argued that the zigzag and staircase with incline paths were more complex than the staircase path. She recognized that there were more corners ("more zigzags") in the zigzag (9) and staircase with incline (12) paths than in the staircase (7), diagonal (0), and L-paths (1). Then, focusing on the zigzag path, she said that the zigzag path instead of moving towards the destination, "slants down" away from it. As a result of the mutually supportive conclusions from applications of the detour and complexity criteria, she decided that the zigzag and staircase with incline paths were longer than the other paths.

This serendipitous association between the detour and complexity criteria did not endure. Afterwards, she also used each criterion individually. In this excerpt from session II, she apparently used the detour criterion without invoking the complexity criterion.

[FF, II, 64-75]

FF: Well, of these [staircase, L, and zigzag], I still think B [zigzag] is longer.
I: Why do you think B [zigzag] is longer?
FF: Well, I just, looks like it, because of the slants.
I: What is it about the slants that makes it longer?
FF: Well, because these [points to one of the staircase's horizontal segments], they're just straight, and this [line segment slanting downward from the left on the zigzag path] is slanted instead of going straight across. Cause if this [zigzag's slant] was straight, then I mean it would be shorter than this line [itself], too. So I still think this [zigzag] is bigger than that [L-path or staircase or both?].

Without referring to the complexity of any of the paths, FF alluded to the "slants" of the zigzag path that contrast with the other path's segments which are "straight across." She also applied the complexity criteria without invoking the detour criteria in the following segment.

[FF, II, 103 - 111]

FF: Hmm, I guess I'll try this one. [moves staircase with incline and L-path transparencies on top of one another] (5)
I: What are you thinking?
FF: D looks longer.
I: Why is that?
FF: Cause it seems to have more (2)
I: More what?
FF: More lines, I guess. I'm not sure, but it [staircase with incline] looks like it's longer because it has more of them [lines?].

Here FF said that the staircase with incline had "more," suggesting a complexity explanation without any references to detours. Hence, the use of both the complexity and detour criteria in one explanation did not appear to form a permanent association between them.

In FF's case, she applied the intuitive concepts separately, but GG seemed to have coordinated them together so that the two criteria interacted with one another in his explanation. In the following excerpt, GG gave a compression explanation demonstrating that the diagonal path was shorter than the staircase path.

[GG, I, 40-46]

I: What do you think about B [diagonal] and C [staircase]?
GG: C is longer.
I: Why is C [staircase] longer than B [diagonal]?
GG: B [diagonal] is a straight line from here [upper left hand endpoint] to here [lower right hand endpoint] and then this [staircase, C] they [the corners of C?] all end up in the same end --same line and then this one [staircase, C] goes waves, so if you stretch it out longer, it'll go about here [point on extension of the diagonal].

GG later elaborates on his first explanation when the interviewer asks him for an alternate solution.

[GG, I, 90-104]

I: Yeah, how else could you explain that B [diagonal] is shorter than C [staircase]?
GG: The same way as I did to explain it to you. Put it together like this [moves B and C transparencies on top of one another] and put it [corner of staircase, C] an equal line [convert the staircase into a linear path]. Where it ends in the same place, they start right here, it ends right there. and then this one [diagonal, B] just goes straight and this one [staircase, C] is going away like this [points to a corner of C], so ah,
I: Can you repeat that? I didn't quite hear that.
GG: C [staircase] goes like way up and down, up and down. B's [diagonal] going just straight line so C [staircase] could be longer than that. You could measure it out like this [pulls lower endpoint of the staircase path out along an extension of the diagonal path] .

GG began by saying that he's explaining "the same way" as he did before and started converting corners into straight line segments. Immediately after referring to the line as an object ("put it ") however, he said that the line as "ends in the same place," "starts right here," and "ends right there," in comparison to the straight path B which "goes straight. " In contrast, the staircase path "is going away." GG elaborated on this apparent detour explanation, saying that the staircase path goes "up and down, up and down" whereas "B's [diagonal] going just straight." Then, he demonstrated the compression argument again by pulling out the staircase. The smooth explanation from one intuition into the other suggests that GG integrated both criteria in his understanding of this path comparison between the diagonal path and the staircase path. Converting each corner into a line segment can be both "pulling it out" and changing the detours into straight roads. GG may have recognized that both explanations of "straightening the corners" were essentially isomorphic. Further research may address this conjecture that through the co-incidence and co-occurence of using two intuitions, a person may have the opportunity to build his or her understanding.

Most students did not utilize supportive intuitions together in subsequent comparisons. However, one student may have noticed the isomorphism between the detour and the compression intuitions. If so, multiple intuitions that support one another provide opportunities for the student to relate the intuitions to one another, thereby building more complex knowledge.

4.4 SUMMARY OF RESULTS

When asked to solve these geometry problems, these students initially used intuitions that originated from everyday experiences. However, they did not systematically apply them to every path comparison. In addition to bouncing from one intuition to another to compare paths, these students demonstrated the fragmented nature of their intuition through their extremely limited applications and their "waffling". Moreover, students produced both correct and incorrect results by using their intuitive concepts. Despite their intuitions' inadequacies, these students continued using them at the beginning of the second session. Thus, their prior intuitions had a higher priority than their co-existing successful competitor, the align & compare method that they would later use to solve the problem.

Some students applied several intuitions to a single problem leading to either conflicts or mutually supported conclusions. Although they typically resolved their conflicts by choosing one intuition over another, the "losing" intuition could still reappear as the criterion for solving another path comparison. Constraints about the transitive property of equalities and partial orders also help resolve conflicts. Most students who found converging intuitions that mutually supported one another simply let them drift apart from their ephemeral, chance encounter. However, one student may have built more lasting connections between two of his intuitions to form a larger piece of knowledge.

5.CONCLUSION

In this paper, I have demonstrated the utility as well as the inadequacies of some intuitions. They are not only "misconceptions" or "learning barriers" (Hawkins et al. 1981), but can be productive elements of knowledge. Instead of being told that particular intuitions (perhaps knowledge in general) have been labeled as correct or incorrect, a student may learn more by assessing them with more sophisticated criteria such as range of applicability, ease of use, and coherence with other ideas.

Intuitions exist in pieces, unlike the theories of experts. Since they have an independent source of reinforcement through everyday activities, they may co-exist with other standard mathematical knowledge despite instructional attempts to uproot them.

What role, then, should intuitions play in learning and development? One possibility is to help students recognize and delineate an intuition's range of application. In addition, students may learn to link an intuition to an appropriate mathematical concept or procedure, so that the intuitive concepts act as pointers to formal mathematics. Furthermore, since these intuitions were useful in some situations, it may be possible to build formal mathematical understanding on the foundations of carefully selected intuitions. "Complexity", for instance, seems to precede quantitative reasoning. Some intuitions seem intimately tied to one another. For example, what is the relationship between "straightness" and the other intuitions: compression, detour, and complexity?

Finally, this study highlights the potential complications of using even simple representations. As this study of simple lines showed, students can use their intuitions to interpret a line as both a string and a moving object. Since students may well use these intuitions to understand the vast number of representations that they face in a mathematics classroom, teachers would benefit from considering these different perspectives. Lines are not just lines, students imbue them with meaning.

REFERENCES

Carey, S. (1985). Conceptual Change in Childhood. Cambridge, MA: MIT.

Carey, S. (1988). Conceptual differences between children and adults. Mind & Language, 3, 167-181.

Clement, J. (1982). Students' preconceptions in introductory mechanics. American Journal of Physics, 50, 66-71.

Confrey, J. (1990). A review of the research on student conceptions in mathematics, science, and programming. Review of Research in Education, 16, 3-56.

Davis, R. B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconceptions Stages. Journal of Mathematical Behavior, 5, 281-303.

diSessa, A. A. (1983). Phenomenology and the evolution of intuition. in D. Gentner and A. L. Stevens, (Eds.) Mental models. Hillsdale, NJ: Lawrence Erlbaum.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman and M. Pufall (Eds.), Constructivism in the computer age. Hillsdale, NJ: Lawrence Erlbaum.

diSessa, A. A. (in press). Toward an Epistemology of Physics. Cognition and Instruction.

Eylon, B. & Linn, M.C. (1988). Learning and instruction: An examination of four research perspectives in science education. Review of Educational Research, 58, 251-301.

Fischbein, E., Deri, M., Nello, M., Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. Journal for Research in Mathematics Education, 16, 3-17

Halloun, I. A., & Hestenes, D. (1985). Common sense concepts about motion. American Journal of Physics, 52, 1056-1065.

Hawkins, D., Apelman, M., Colton, R., Flexner, A. (1982). A report of research on critical barriers to the learning and understanding of elementary science. Washington, DC: National Science Foundation.

Hewson, P.W. (1981). A conceptual change approach to learning science. European Journal of Science Education, 3, 383-396.

Johnson, M. (1987). The body in the mind: The bodily basis of meaning, imagination, and reason. Chicago: Chicago.

Kuhn, T. S.. (1965). The structure of scientific revolutions. Chicago: Chicago.

Lakoff, G. (1992). The contemporary theory of metaphor. Unpublished manuscript. University of California, Berkeley.

Mason, M. M. (1989, March). Geometric understanding and misconceptions among gifted fourth-eighth graders. paper presented at the Annual Meeting of the American Educational Research Association. San Francisco, CA.

McCloskey, M. (1983). Naive theories of motion. In D. Gentner & A. L. Stevens (Eds.) Mental models. Hillsdale, NJ.

Nersessian, N., & Resnick, L. (19??). Comparing historical and intuitive explanations of motion: Does "naive physics" have a structure? Cognitive Science, 41, 1123-1130.

Nesher, P. (1987). Towards an instructional theory: The role of students' misconceptions. For the Learning of Mathematics, 7(3), 33-40.

Newell, A. & Simon, H. (1972). Human problem solving. Englewood Cliffs, NJ: Prentice-Hall.

Ogborn, J. & Bliss, J. (1990). A psycho-logic of motion. European Journal of Psychology of Education, 5, 379-390.

Piaget, J., & Inhelder, B. (1967). The child's conception of space. (F. J. Langdon & J. L. Lunzer, Trans.). New York: Norton. (original work published 1948).

Piaget, J., Inhelder, B., & Szeminska, A. (1960). The child's conception of geometry. (E. A. Lunzer, Trans.). New York: Norton. (original work published 1948).

Piaget, J., & Szeminska, A. (1952). The child's conception of number. (C. Gattegno & F. M. Hodgson, Trans.). London: Routledge & Kegan. (original work published 1941).

Rogoff, B. & Gardner, W. (1984). Adult guidance of cognitive development. In B. Rogoff & J. Lave (Eds.), Everyday cognition. Cambridge, MA: Harvard.

Shaughnessy, J. M.. (1977). Misconceptions of probability: An experiment with a small-group, activity-based model building approach to introductory probability. Educational Studies in Mathematics, 8, 295-316

Skinner, B.F. (1968). The technology of teaching. Englewood Cliffs, NJ: Prentice-Hall.

Strauss, A. L. (1987). Qualitative analysis for social scientists. Cambridge: Cambridge.

Talmy, L. (1988). Force Dynamics in Language and Cognition. Cognitive Science, 12 , 49-100.

Vinner, S. (1990) Inconsistencies: Their causes and function in learning mathematics. Focus On Learning Problems in Mathematics, 12, 85-97.

Vygotsky, L. (1978). Mind in society: the development of higher psychological processes. (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds. & Trans.) Cambridge, MA: Harvard.

Vygotsky, L. (1986) Thought and language. (A. Kozulin, Trans.). Cambridge, MA: MIT Press. (original work published 1934).

Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. Journal of Child Psychology and Psychiatry, 17, 89-100.

Wood, D., Wood, H., and Middleton, D. (1978). An experimental evaluation of four face-to-face teaching strategies. International Journal of Behavioral Development, 1, 131-147.

Appendix A: Coding Scheme

Compression

Unfolds a path. The path is compressed so it's actually longer than it seems. Think of the line as a string and pull it out.

Detour

A path is doing something else instead of going to the destination. Wanders away from the destination instead of moving towards it. The path detours or goes out of the way. A path turns and wastes space.

Complexity

A path has more of something, like lines, and is longer. Complex paths have more components and are therefore, longer.

Straightness

The straight line path is the shortest. A particular path is straighter than another path and is therefore shorter.

Analysis by Parts

Subtypes

A's vertical segment is (almost) as long as B.

convert the paths into straight lines & compare

convert each part into a number (measure) & add

match (by rotating) and see what's left over

align corresponding horizontal and vertical segments with one another and see what's left over

V Visual

It looks like it.

O Other

Appendix B: Instruction Script

[...] indicates a possible student response.

(Take out graph paper pad, string, scissors, tacks, 5 index cards, two markers, a pencil, and an eraser)

You can use some of these materials to check your answers. Please tell me what you are thinking as you're working.

[Student measures and gathers data]

> (If the student doesn't write down her data, say "You may want to write that down. You probably don't want to do it again if you forget")

(If the student isn't trying to be precise, the lengths were chosen so that measurement errors tend to cancel.)

> How do you explain this result?
>
> [I don't know.]
>
> > Can you change C so that it looks like A?
> >
> > Do you see any relationship between the parts of A (L-path) and the parts of C (staircase path)?
> > [No.]
> >
> > > How do the vertical, up-down segments compare?
> > > [I don't know.]
> > >
> > > > How long are they?