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MISCONCEPTIONS IN LEARNING MATHEMATICS

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I have come across many students for a decade while teaching, showing some disinclination/aversion to learn mathematics due to the simple reason that the subject was either not well presented or wrongly presented. Some of them have shown remarkable progress when once they understand the meaning,the language, symbols and theinter-connections among the words, topics etc. The mathematician's language is distinct from the ordinary language. A shift in emphasis from **Memorisation** to **Meaning**, **Computation** to **concepts, the what** to **the why & how** will enable students to avoid the mistakes, carry out the logical operations correctly and solve the problems with ease and confidence. I have attempted to classify the general **misconceptions** into four categories A,B,C & D. I have tried to give illustrations to justify how the misconception has occured and to seek remedial steps. These categories are:

A. Pair of words which appear to be the same but conceptually different in Mathematics.

B. The use of a mathematical word as an adjective in different contexts giving different interpretations.

C. The use of same word in different meanings.

D. The use of same symbol to indicate different objects /operations

Let me give some examples for the above.

[A]

1) Number and numeral:

These are used synonimously while teaching / learning Maths. A number is an idea while a numeral is a written expression of that idea. We compare numbers and not numerals. But, if we fail to take note of the difference between a number and a numeral, we may come into trouble.

For example, 1/3 = 2/6 as absolute numbers.

This does not mean that we can replace 2/6 or 1/3 by the other,at will. Let us see this example.

$$\sqrt[3]{-8} = (-8)^{1/3} = -2 \quad \text{.... I}$$

Also $\sqrt[3]{-8} = (-8)^{1/3} = (-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$ II

Thus $\sqrt[3]{-8}$ is either -2 or +2

Is -2 = +2 ??

Where is the mistake ?

The mistake is while replacing 1/3 by 2/6 only. Why ?

Here (-8) is the number to be operated and 1/3 is the Operator (exponent), Which is nothing but a symbol for a command `Take thecube root of'.

2) Fallacy and Paradox :

An incorrect reasoning coupled with an apparently logical explanation justifying the result obtained is a fallacy. In mathematics, it is the incorrect chain of reasoning essential to the situation that brings out a fallacy.

Let us assume 1 = 2

then 2 = 1 (symmetric property of equality)

3 = 3 (Adding LHS and RHS)

But 3 = 3 is true

$\therefore$ 1 = 2, which is our original assumption

But 1 ≠ 2

A paraadox is a statement which appears to be true when it is actually false and appears to be false, when it is actually true.

Let me take an example

" A loving husband tells his wife that she will receive an unexpected gift for her birthday. It will be a gold watch. Does he really give his gift ?

The husband sets three conditions :

i) The gift will be given on her birthday
ii) The gift will be unexpected.
iii) The gift will be a gold watch.

As the husband has promised to give a gift on her birthday, of course,he does. But as the gift will be a gold watch , it is already expected and so the husband does not give a gift to his wife because, the gift (which is a gold watch) should be unexpected.

Let us take an algebric example.

if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$ (Invertendo)

b d a c

Suppose we use this rule to change

$$\frac{x-3}{x-1} = \frac{x-3}{x-2} \quad \textbf{into} \quad \frac{x-1}{x-3} = \frac{x-2}{x-3}$$

Are these expressions equally meaningful ?

If $\frac{x-3}{x-1} = \frac{x-3}{x-2}$, we can say x-1 = x-2

or -1 = -2

or 1 = 2 ?? or x-3=0 => x=3

If $\frac{x-1}{x-3} = \frac{x-2}{x-3}$ here also x = 3

But 3 is not a solution of the above equation .

The truth is that, $\frac{x-3}{x-1} = \frac{x-3}{x-2}$ has a perfectly good solution but

$\frac{x-1}{x-3} = \frac{x-2}{x-3}$ has **no** solution.

If it has a solution , it will be x=3. when you replace x by 3,

we get $\frac{3-1}{3-3} = \frac{3-2}{3-3}$

i.e.

$\frac{2}{0} = \frac{1}{0}$ Which is not a permissible operation

The problem can be generalised to note that

$\frac{x-a}{x-b} = \frac{x-a}{x-c}$ with $b \neq c$ **always** has the solution x = a while

$\frac{x-b}{x-a} = \frac{x-c}{x-a}$ with $b \neq c$ **never** has a solution.

3) Obvious, elegant and Trivial :

The words 'Obvious' , 'Elegant' and 'Trivial' are often used by us , but interchangebly. This is not correct.

Something is obvious , if anyone would agree, that it is so. For example, two st. lines intersect at only one point. Although this can be automatically proved, yet it is taken as obvious. A proof is trivial if it is already obvious , before pronouncing the judgement; for example , 1 is the factor (or exact divisior) of every number.

A proof is elegant , if what it demonstrates was not obvious before the demonstration but becomes thereafter. For example let us solve the equation

$$1/a + 1/b + 1/x = 1/a+b+x$$

Instead of clearing off fractions and then attempting to solve the problem, We note that the problem will reduce itself to a quadratic and therefore , at the most , have two solutions only. By inspection, let us put x = -a and try. We see

$$LHS = 1/b = RHS$$

$\therefore$ x = -a is a solution

similarly, x = -b is another solution

-a , -b are the only two solutions to this problem.
Such a solution is called elegant.

4) Undefined and NOT defined :

Undefined objects in maths exist. Points, planes, Numbers etc are undefined. But not-defined are Non-existent. The slope of Y-axis is not defined (as Tan 90 = 1/0 is not defined). The x-intercept of a line parallel to the x axis is not defined.

5) No and Zero :

These are not the same ideas in maths. A boy who was absent for an examination will be given no mark whereas, a boy, who has answered all the answers wrongly would get 0 mark.

A vertical line has NO slope (or gradient) while a horizontal line has zero slope. The number of elements in a void (or Vacauous) set is 0 but the void set has No elements.

6) Exactness, Precision and accuracy :

The exact value of $\sqrt{2}$ is $\sqrt{2}$ whereas 1.414 is an accurate value for $\sqrt{2}$. Exactness means neither more nor less but unique in value. Precision and accuracy are two different ideas used in measurement . Accuracy depends on the number of significant digits used in measurement while precision depends on smaller measurement.The smaller the measurement, the greater the precision.

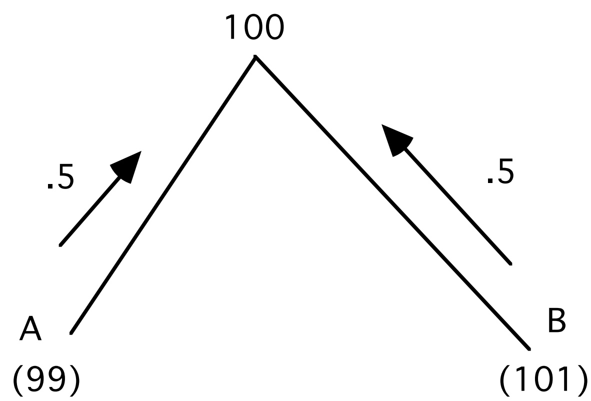
For example , 5104 m, 51.04 m, 0.5104 m are all equally accurate but the last one, viz, 0.5104 m is more precise as it has a least unit of measurement (0.0001)

The accuracy of a measurement depends upon the number of significant digits in its value, related to the relative error in its value. The smaller, the relative error, the greater the accuracy. Suppose we write the figure 4.67 . We usually take this to be correct to 3 significant figures . If this figure was obtained correct to 0.001, then it should be written as 4.670, but unfortunately this rule is not followed.

4 and 4.0 are equal if they are absolute, i.e, if they are pure numbers,but, they are distinctly different if they represent measurements. i.e, any no. between 3.5 and 4.5 is represented by 4 whereas, those between 3.95 and 4.05 is represented by 4.0.

What is the value of $\sqrt{64.000}$?
It is not +8, not +8.0, not +8.00 .
Its value is 8.0000 (precisely)

Suppose A & B have each measured the length of the hall and asserted 99 mts. and 101 mts. respectively. When the hall is certified to be of length 100 mts., who is more accurate to the correct (certified) measurement?



The absolute error in measurement (i.e, the difference between the measured value and the true value) of both A & B is 0.5.
relative error of A is 0.5/99 and of B is 0.5/101

Let us compare the fractions 0.5/99, 0.5/101
i.e. 5/990, 5/1010
i.e. 1/990, 1/1010
clearely 1/990 > 1/1010 (as $\frac{1010 > 990}{990 \times 1010}$)

Thus the relative error of A is more than that of B.
i.e. A has made more error in measurement.
i.e. B has made less error in measurement.
i.e. B is more accurate than A.
(B's measurement contains three significant figures while A's only two)
Does this have any connection ?

7) Errors and Mistakes:

' To err is human'
'An error does not become a mistake if it is corrected!
'Errors like straws, upon the surface flow,
He who has to search for pearls, must dive below'

Errors are inaccuraciesand these need not / cannot be corrected , even if detected. They are caused by inexact information such as the data error or the method error such as we agree to write
$\sqrt{2}$ =1.414,
∏=3.1416 or 22/7, e=2.71838, Sin x=x=Tan x for small x,$(1+x)^n$=1+nx etc.
Sometimes we truncate a decimal or round off the digits after the decimal point , thus causing errors.

Mistakes are those which can be corrected immediately when detected. This can happen while copying numbers, using wrong rules or formula carrying decimals and operations such as additions, multiplications etc. reading tables and so on. These can be avoided by taking some precautions such as the neat or orderly setting out of calculations,use of checks and exercising care in reading tables, copying numbers carefully etc.

8) Factor and Divisior :

All divisors are factors but every factor need not be a divisor . For eg. , zero factors are not divisions.
Consider this example.

	Dividend Polynomial	Divisior	Quotient	Remainder
i)	x^3-xy^2-8xy^2+10y^3	x+3y	x^2-4xy+4y^2	-2y^3
ii)	10y^3-8yx^2-yx^2+x^3 2x^3/27	3y+x	(10/3)y^2-(34/9)yx+(25/27)x^2	
iii)	x^3-xy^2-8xy^2+12y^3	x+3y	x^2-4xy+4y^2	0
iv)	12y^3-8yx^2-yx^2+x^3	3y+x	4y^2-4yx+x^2	0

What do we see in the above?

The quotient is x^2-4xy+4y^2 in (i)

$(10/3)y^2-(34/9)yx+(25/27)x^2$ in (ii)

although the dividend and the divisor are the same in (i) & (ii)

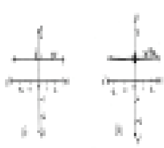
but written in the reverse order.

We also get distinct remainder. But in case (iii) and (iv), the quotient and the remainder are the same, although we have written the dividend and divisor in the reverse order and performed division operation . Why ? In case (iii) and (iv), the divisor is an exact factor of the dividend whereas in case (i) and (ii), it is not.

9) Continuous and Continual :

We very often use these terms wrongly. It is very common to hear the complaint from a teacher. "This boy is disturbing me continuously". `Continuous' means `no break in between' while `Continual' means ` occasional breaks in between'. We have to say that `This boy is disturbing me continually' as the boy cannot go on disturbing continuously (without a break)

Let me take an example.
Draw the graph of the curve y=1 and y=x/x



y=1 is a st. line, parallel to the x axis (at a distance of 1 unit from the x axis) while y=x/x is a st. line, parallel to the x-axis(at a distance of 1 unit from the x axis) but with a hole or a puncture on the y axis . In other words, there is no value for y corresponding to x=0 as 0/0 is not permissible and thus y=x/x is a 'broken' line as distinct from y=1. This is 'Continual' or 'discontinuous'.

10) Cost and Price :

These words are also used interchangebly. We also ask 'What is the cost price' ?

In fact 'Cost' goes to the value, which the shopkeeper has paid in order to get the object, Whereas 'Price' goes to the value, which the buyer has to pay to the shopkeeper to purchase that object. 'What is the price of this watch' is the correct question if one intends to buy it.

11) Commutative property and symmetric property :

Commutative property is defined with respect to an operation such as 'addition', 'multiplication' etc.

We say $a+b = b+a$

$a.b = b.a$

(because of commutative property of addition and multiplication respectively).But symetric property is defined with respect to a 'relation'.

if $a = b$ then

$b = a$

Equality has symetric relation or equality has symetric property.
Similarly, if $l \mid\mid m$, then $m \mid\mid l$

(by the symetric property of parallelism).

12) Middle and Centre :

These are used very much interchangingly. In fact, this causes confusion in ideas. We ask, 'Which is the middle number in the series: 11,12,13,15,16,19 ? Find the middle term in the expansion of $(2x+3y)^7$ and $(2x+3y)^8$. What do you notice?
Is 'middle' the 'centre' always ?

A centre is an exact location , such as, the centre of a circle. The middle is a representative attribute of anything which is in the neighbourhood of the centre. See this example

Find the median of 10,12,15,16,19,21
The median x is defined here as

$15<x<16$

(It is very common to take the median as $(15+16)/2$ i.e. 15.5)

13) Identities, Equations and Inconsistencies :

Mathematical sentences can be classified into three categories.

i) Identities **:-** These are sentences which are always true.
eg. $5(x-2)+9 = 5x-1$

ii) Equations **:-** These are sentences which are not oftentrue; which are sometimes true.
eg. $\frac{1+\cos X}{1-\cos X} = \text{cosec } X + \text{Cot } X$

iii) Inconsistencies :- These are <u>never</u> true.

eg. $5 \sin X = 20$

14)

Many more examples of the above can be given which createa confusion in concepts owing to wrong understanding or loose understanding in meaning. For eg. 'opposite of a number and 'reciprocal of a number'.

[B]

Sometimes, the confusion in meaning is caused by the usage of the same word in different contexts and in totally unrelated situations.
For eg., We say/use <u>'Conjugate '</u> in many places.

i) Conjugate points, Conjugate lines, Conjugate diameters, Conjugate hyperbolas, Conjugate binomial surds, Conjugate complex numbers, Conjugate imaginary points, Conjugate imaginary lines.

Conjugate points :- Two points are conjugate with respect to a conic section if each lies on the polar of the other.

Conjugate lines :- Two lines are conjugate with respect to a conic section if each passes through the pole of the other.

Conjugate diameters :- A pair of conjugate lines that pass through the centre of a conic section.

Conjugate hyperbolas :- Two hyperbolas for which the transverse axis and the conjugate axis of one are the conjugate and the transverse axis of the other. Their equations are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Conjugate binomial surds :- $\sqrt[a]{b} + \sqrt[c]{d}$ and $\sqrt[a]{b} - \sqrt[c]{d}$ where a,b,c,d Q and b , d are not both rationals.

Conjugate complex numbers :- a+ib , a-ib with product $a^2 + b^2 \in R$

Conjugate imaginary points:- The pair of points for which a circle intersects a st. line which has perpendicular distance from the centre which is greater than radius.

Conjugate imaginary lines:- If $h^2 < ab$ in $ax^2 + 2hxy + by^2 = 0$,
the two lines formed out of them are a pair of conjugate imaginary lines.

(ii) The word 'Base' is used in many places. Base of a logarithem, base of a triangle, base of solid, base in Trignometry.

(iii) Similarly we have Absolute value of a real number, Absolute value of a vector, Absolute term in algebra, Absolute error (in measurement)

(iv) We also use the word 'Null' in different contexts. Null circle, Null Ellipse, Null Matrix, Null set, Null sequence, Null vector.

[C]

Sometimes, a Single word may mean many objects in mathematics. For example,

(i) Tangent: A Geometrical object.
A trignometric ratio.

(ii) Median: The line joining the mid point of a side and the opposite vertex in a triangle.
or
The line joining the mid points of non-parallel sides of a trapezium.
or
The mid-value of a series arranged in ascending order (statistics).

(iii) Dividend: The number which is divided.
The company declares profit at the end of a financial year. This profit is called dividend.

(iv) Cube: A rectangular solid having six equal square faces.
The product obtained by taking a number three times as a factor to raise to third power.

[D]

Sometimes, the same symbol may create confusion in concepts.

For eg.

i) : Colon; Ratio; Such that;

ii) . Period; Full stop; Into; Decimal point; Dot product of two vectors.

iii) --> Arrows; Indicating directions; Steps in solutions; Vectors

iv) X Wrong symbol; Multiplication; Vector product; Negative command in an ordinary sense such as 'do'nt' as 'X' mark in the bottle of poison etc.

v) Δ Delta; Triangle; Determinent of matrix; Discriminent of a quadratic equation.

Mathemeticians are virtually concerned with accurate transmission of information. This often means two things.

i) <u>one is precise statement of definitions and</u>

ii) theother is the <u>adoption and agreed upon colloquialisms</u> tobe used, where their meaning cannot be misunderstood. Failure to recognise the first requirement results in <u>poor mathematics</u> while that of the second results in <u>poor mathematics teaching</u>. Mathematics derives its ideas from hunches, intuition, analogies, conjectures guess and experiments for which language is vital. Therefore misconceptions can be avoided if the linguistic aspect of maths is not ignored in the teaching-learning process of mathematics.

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