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In our joint work in both mathematics and science education, it became apparent that many of these alternative conceptions are based on the same intuitive rules. We have so far identified two such rules: "The more of A the more of B", and, "Everything can be divided by two". The first rule is reflected in students' responses to many tasks, including all classical, Piagetian conservation tasks (conservation of number, area, weight, volume, matter, etc.), in all tasks related to intensive quantities (density, temperature, concentration, etc.), and in tasks related to infinite quantities. The second rule is observed in students' responses to tasks related to successive division of material and geometrical objects and in successive dilution tasks. In this paper we shall describe and discuss the first rule and its relevance to science and mathematics education.

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# Intuitive Rules in Mathematics and Science: The Case of "More of A-More of B"

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Throughout the past 20 years, researchers have studied students' conceptions and reasoning in the context of science and mathematics. Most of this research has been content specific. It has been found that students hold alternative ideas which are often not in line with the accepted scientific frameworks (Driver, Guesne, & Tiberghien, 1985; Eylon & Lynn, 1988; Fischbein, 1987; Hart, 1981; Osborne & Freyberg, 1985; Perkins & Simmons, 1988).

In our mutual work in both mathematics and science education, it has become apparent that many alternative conceptions are based on common, intuitive rules. We have so far identified two such rules: "The more of A, the more of B," and "everything can be divided by two." In this paper, we shall describe and discuss instances of the first rule.

## CONSERVATION TASKS

Conservation refers to the understanding that quantitative relationships between two objects remain invariant (are conserved) in the face of irrelevant perceptual deformations of one of the objects. In the standard conservation task, the subject is first shown two objects that, in addition to being perceptually identical, are known to be equivalent with respect to a certain, quantitative property. The experimenter then proceeds to deform one of the objects in such a way that the perceptual identity is destroyed while the quantitative relationship is maintained. The subject is asked whether the objects are still quantitatively equivalent.

We shall now consider some examples of students' responses to scientific and mathematical conservation tasks.

A. SCIENTIFIC TASKS

1. Conservation of Quantity of Matter

Piaget and his colleagues' studies on the development of the conservation of quantity of matter principle in children's minds are concerned with their dependence on the shape of a given object when asked to judge the relative quantity of matter. When asked to compare the (equal) amount of water in two different shaped cups, children up to the age of about five or six, paid attention only to the relative heights of the water in the two arguing that "there is more water in the taller cup" (Piaget, 1965/1952). In a similar experiment involving the conservation of clay, children's responses were split. some children argued that the thicker disc has more clay than the thinner one." Other children claimed that "the longer piece of clay has more clay than the shorter one" (Piaget & Inhelder, 1974/1941).

2. Conservation of weight

(a) Translocation. Children, in another study, were presented with two identical balls of clay (Piaget and Inhelder, 1974/1941). One of the balls was transformed into a coil, and the child was asked whether the two objects were still equally heavy. It was found that some young children (ages seven and eight) claimed that the coil was heavier because it was thicker. Other children at the same ages claimed that the coil was heavier because it was longer.

(b) Changes of state. Stavy and Stachel (1985) studied children's understanding of the invariance of weight during the process of solid to liquid state transformation. They presented children in ages 5 to 15 with two identical candles, one of which was then melted. The child was asked about the equality of weight (whether the solid candle and the melted one weighed the same). It was found that many children between the ages of six to ten argued that "the solid candle weighs more because it is harder or stronger than the liquid candle."

(c) Expansion. Megged (1978) studied children's understanding of the invariance of weight during the process of heating water. In this case, the volume of the heated water is actually larger than that of the unheated water, but their weight remains constant. Many children ages six to ten, however, argued that “the heated water is heavier because its volume is larger.”

3. Conservation of Volume

Lovell & Ogilvie (1961) asked children to compare the amount of water spilled if two cubes with exactly the same shape and size, one of which was made of wood and the other of lead (and therefore much heavier), were lowered very carefully into a full pint pan. They found that many of the children between the ages of six and ten thought that a heavier cube displaces more water than a lighter cube of the same size and shape.

B. MATHEMATICAL TASKS

1. Number conservation.

Piaget’s (1965/1952) studies on the development of the concept of numbers are concerned with children’s dependence on length and density information when they are asked to compare the number of objects in two parallel rows, containing the same number of objects. Young children (until the age of about five to six) only paid attention to the relative length of the two rows. Rows of the same length were said to be equally numerous; otherwise, the longer row was said to be more numerous than the shorter row. Some older ages, based their judgments on the relative density of the two rows, stating that the denser row was more numerous.

2. Conservation of area.

Piaget, Inhelder, and Szeminska (1960) presented children with two identical rectangular arrangements of six blocks. One arrangement was then altered by removing two blocks from one end and placing them on the other. The children were then asked if the rectangles were still the same size and “had the same amount of room.” They found that many children aged five to six tended to argue that one configuration was larger because it looked longer.

3. Conservation of angle.

Noss (1987) and Foxman & Ruddock (1984) presented children with two identicial angles, one of which had longer arms than the other. They found that many children between the ages of ten and fifteen argued that "the angle with the longer arms is bigger."

In all these cases, as well as in other cases of conservation, children responses seem to be derived from a general rule according to which "more of A implies more of B." A similar phenomenon was observed with tasks related to intensive quantities.

**INTENSITIVITY TASKS**

Intensive quantities are formally defined as follows "If a,b,c,... are parts of a system, and y is a property such that y(a) = y(b) = y(c)..., then y is said to be an intensive quantity. The value of y for the entire system may be defined by y(sys) = y(any part). Clearly y(sys) is independent of the size (or extent) of the system." (S.G. Canagaratna, 1992, p. 957).

Intensivity tasks involve presenting the subject with two systems which are identicial with respect to a certain intensive quantity but different in size. The subject is asked to judge or to compare the values of the intensive quantity in the two systems.

A. SCIENTIFIC TASKS

1. Concentration

Stavy, Strauss, Orpaz & Carmi (1982) presented children with three cups of sugar water of the same concentration. The contents of water from two of these cups were poured into an empty cup, and the children were asked to compare the sweetness (concentration) of the combined cup with that of the original third cup. It was found that the majority of children ages six to ten claimed that the combined cup was sweeter. Two types of justifications were presented: "the one with more sugar is sweeter" and "the one with more water is sweeter". Both of these justifications share the structure of "the more of A, the more of B." In this case the perceptual change in the quantity of water was salient, and directly elicited "the more water, the sweeter" response. Most likely, this salient difference in the amount of water also indirectly encouraged "the more sugar, the sweeter" response. The reasoning behind this response was probably that "the more water, the more sugar," therefore "the more sugar, the sweeter".

2. Temperature

Strauss, Stavy and Orpaz (1977) presented children with three cups containing equal amounts of hot water at the same temperature. The water from two of the cups was poured into an empty cup, and children were asked to compare the temperature of the water in the combined cup with that of the third cup's contents. The majority of children ages six to eight claimed that the water in the combined cup was warmer. These incorrect judgments were justified with reference to the amount of water -- namely, "the more water, the warmer". A similar response was observed with older students, who were asked to refer to the same problem, and presented with numerical values of temperature (e.g., 30˚ c in each cup). Most children between the ages seven and eleven argued that the temperature of the mixture was higher than that of the third cup. These responses were accompanied by an arithmetic calculation (e.g., 30+30=60).

B. MATHEMATICAL TASKS

1. Comparing infinite sets

Students aged 13 to 25 were asked to compare the number of points in two line segments. Even though, according to Cantorian set theory, any two line segments contain the same number of points, about a half of the students in each grade level claimed that "the longer line segment contains more points". This response was provided by adults as well (Tirosh, 1991).

Students' reactions to these as well as to other intensivity tasks seem to evolve from the application of the same general rule "the more of A, the more of B". This response was observed in both children and adults.

## OTHER TASKS

Additionally, the intuitive rule "the more of A, the more of B" seems to operate in many other instances other than those involving conservation or intensivity tasks. We shall provide few examples.

A. SCIENTIFIC TASKS

1. Free fall

Gunstone & White (1981) presented first-year physics students with a problem: an iron sphere and a plastic sphere of the same diameter were held next to each other, two meters, above a bench. The participants were asked to compare the time it would take for the metal sphere to fall to the bench compare with the time it would take the plastic sphere. Some of the students claimed that the metal sphere would fall faster because "a bigger weight will cause bigger acceleration".

2. Dilution tasks

Stavy, Strauss, Orpaz and Carmi (1982) presented children aged four to eight with two cups of sugar water of different concentrations: One cup was full of water and one teaspoon of sugar was added and mixed. The other cup was half full of water and one teaspoon of sugar was added and mixed. The child was asked whether the sweetness of the water in the two cups was the same or different, and if different, in which cup the water was sweeter. Many four to eight year-old children produced incorrect judgments, claiming that "the cup with more water is sweeter". In this case, children were treating the inverse function as if it were a direct function: that is, more water means sweeter.

B. MATHEMATICAL TASKS

1. Multiplication always makes bigger

Tirosh, Wilson, Graeber, & Fischbein (1993) presented college students with several tasks concerning the relationship between factors and products in multiplication expressions. Many of these students argued that when one of the factors is increased, the product always increases. While this statement holds for natural numbers, it does not hold for negative numbers [e.g. $2 \times (-4) \geq 8 \times (-4)$]. It seems that the mistaken students were operating according to the rule "the bigger the factor, the bigger the products".

2. Percentage tasks

Rachmani (personal communication) presented students at different age levels with several problems in an attempt to assess their understanding of percentages. One of these problems was the following: "Joe saved 25% of his salary. Maya saved 50% of her salary. Can you determine which of them saved more money?" Obviously, the answer to this problem is "no", because the salaries of Joe and Maya are not know. Yet many students responded that "Maya saved more money, because she saved a higher percentage."

**DISCUSSION**

So far, we have presented students' responses to a variety of different tasks that relate to different subject matters -- some deal with physical objects while others refer to mathematical ones, some focus on extensive quantities while others involve intensive one. All these tasks, though, share some common features. In each of them, two objects (or two systems) which differ in a certain, salient quantity are described. The student is then asked to compare the two objects or systems with respect to another quantity. In all these cases, a substantial number of students responded according to the rule "the more of A (the salient quantity), the more of B (the quantity in question)".

In all the cases described above, the response "the more of A, the more ob B" led to inappropriate answers. In many tasks embedded in everyday life and in scientific situations, however, a response based on this rule leads to conclusions that are accurate. For instance, "the more money you have, the more candy bars you can buy".

Thus, clearly, people of different ages tend to use this rule in many situations, in some of which it is applicable and in others of which it is not. The question which naturally arises, then, is why the use of this rule so far outspans its applicability.

One possible explanation is that this rule is used as a first approximation in solving a certain problem when no other information is available. Such approximation may serve as the starting step in a process of exploration. Often, approximation is made in the absence of a solid understanding of the situation. An example of such behavior was the reaction of a science educator from our department when asked to judge whether two given angles were equivalent. She explained that she was unfamiliar with this problem, and therefore it seemed to her that the most sensible answer should be "the angle with longer arc is larger".

A second possible explanation is that the use of this rule gives the solver a sense of understanding the situation, as it creates a casual/logical relationship between the various components of the system. Such a relationship gives the sense of an ability to predict (though often incorrectly).

There are several possibilities as to the nature of this relationship: (a) A qualitative relationship -- as A increases, B also increases. (b) A quantitative relationship of the form B=kA. It could also be that in some cases the relationship between A and B is perceived as qualitative and in others as quantitative.

It was also observed that the solver often views the legitimacy of this rule as self-evident, and that its use is often accompanied with a sense of confidence. Self-evidence and confidence are two major characteristics of intuitive reasoning (Fischbein, 1987). In fact, this rule has some additional characteristics of intuitive reasoning: immediacy, globality and coersiveness. Thus, this rule seems intuitive, yet there is still a need to explain why this intuitive rule is activated in the situations described above. Let us recall that in each of these tasks, the two objects or systems differ in one particular quantity. Our claim is that the intuitive rule is directly activated by immediate perceptual differences or by salient differences between symbols associated with perceptual images, such as numbers. The perceptual differences are often visual, as in the conservation and intensivity tasks. A source of support for this claim is provided by the results of Bruner's (1966) experiment related to conservation of quantity of liquid. In this experiment, children aged four to seven were verbally asked about the conservation of liquid. The task was carried out behind a screen. It was found that half of the four-year-olds and all of the others correctly said that the amount of water was the same when the screen was used. When no screen was present, though, most four- and five-year olds said that "the higher the water, the more water to drink".

We have suggested so far that "the more of A, the more of B" is an intuitive rule which is activated by specific, perceptual input. What could be the origin of this intuitive rule?

At this stage we can only suggest several speculative possibilities.

(a) It is an innate, intuitive rule. The following, surprising excerpt, taken from Tinbergen's (1951) book on the study of instinct is in line with this possibility.

"Oystercatchers preferred a clutch of 5 eggs to the normal clutch of three. Still more astonishing is the oystercatchers' preference for abnormally large eggs. If presented with an egg with normal oystercatcher size, one of herring gull's size, and one doubles the (linear) size of herring gull's egg, the majority of choices fall upon the largest egg" (p.45). It seems that the oystercatchers' decisions are determined by the implicit laws of "the more, the better" and "the larger, the better".

(b) Overgeneralization from successful experiences. As mentioned above, in many instances, both in everyday life and in school situations, the rule "the more of A, the more B" is applicable. Children, starting from early infancy, encounter many situations in which perceptual differences between two objects go hand in hand with quantitative parallel differences in another property of these objects. It is reasonable to assume that children generalize these experiences into a universal maxim: "the more of A, the more of B".

At this stage in our inquiries, it is impossible to determine which of these possibilities is the source of this intuitive rule. Yet in both of the suggested possibilities, repeated experiences that reinforce this rule enhance its use in other seemingly similar situations.

We have mentioned before that this rule is often used in situations in which it is not applicable. Yet, with regard to each of the tasks described above (and to other tasks as well), children at different ages and/or with different levels of instruction at some point start using this rule selectively.

Most four-year-old children, for instance, argue that "the higher the water level, the more water there is" in the conservation of the quantity of matter task, while practically all ten-year-olds know that the amount of water is preserved in this task. How can such changes in response be explained? Why do people use this rule in certain situations and discard it in others?

With age and/or instruction, schemes, rules, and bodies of knowledge related to a specific task and group of tasks are developed or reinforced. At the same time, people realize the inappropriateness of this rule in these tasks. Consequently, in respect to these specific tasks, the rule "the more of A, the more B" loses its impressing power in favor of other, competing knowledge. For instance, in the case of conservation of quantity of matter, the use of the rule "the higher, the more" is replaced by identity or compensation considerations. In addition, with age and/or instruction, children become aware of the need to examine their initial responses, to consider other factors which might be relevant to the task, and to avoid conflicting arguments. Consequently, they gradually learn the boundaries within which the rule "the more of A, the more B" is applicable.

Although at certain ages children cease to use this rule in specific instances, it seems that the rule itself does not disappear but continues to dominate in various other situations. In fact, in many of the instances described in the paper (e.g., comparing segments, comparing angles, free fall) older children and adults kept using this rule when solving certain, given tasks. Thus it seems that like many other intuitive rules, this rule persists and retains its dominancy in certain situations.

It is, however, not clear what the status of this rule is with regard to the specific tasks in which children and adults seem to overcome its effects and give judgments not in line with it. For instance, when a child gives correct judgments to conservation tasks, is it because the rule "the more of A, the more B" ceases to exist for these tasks, or does it exist but fail to compete with other bits of knowledge? A source of support for the possibility that the intuitive rule continues to exist comes from the following example. A very distinguished physics professor, an expert in astronomy, said on a television science program that "the force that the larger star exerts on its moon is larger than the force that the smaller moon exerts on the star". Of course this professor was familiar with Newton's third law (the law of action and reaction), according to which any two bodies exerts the same force on each other. The next day, he explained that under the pressure of the television program, he was not alert enough and gave an immediate uncontrolled response. Such situations are very common.

What is the relevance of such intuitive rules to science and mathematics education? First, it is crucial that teachers, policy makers, and curriculum developers be knowledgeable about the common roots of many seemingly unconnected, content-specific alternative conceptions. Such knowledge has predictive power: it enables teachers and researchers to foresee possible inappropriate students reactions of the type "the more of A, the more of B".

It is equally important to better understand how children overcome the coerciveness of this rule. Such knowledge can help teachers and curriculum developers plan sequences of instruction which consider the role of this rule and the ways to overcome it. Finally, this paper points strongly to the importance of raising students' self-awareness, critical thinking, and their awareness of the need to be consistent. All of these are elements necessary to the formation of boundaries with regard to the use of intuitive rules.

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