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THE INTUITION OF THE PRODUCT

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26 May 1993.

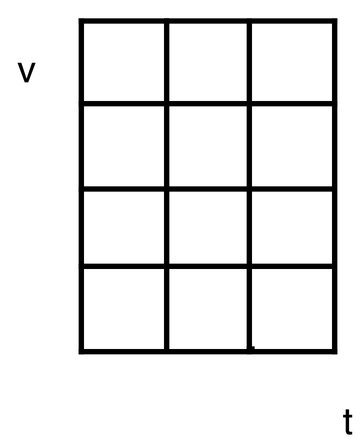
ABSTRACT.

Does multiplication require insight or is it algorithmically defined, in other words, does the product conception rest on experience or on structure? This problem is introduced on the basis of the representation of the covered distance as an area. The second part contains an historical outline of the anontological process of the product conception, that is, the development from object to activity. Intuitions of more than two millennia played a double negative role in the development of the thinking about multiplication. Firstly one could neither correctly imagine the product, nor the proportion, of magnitudes of different kind. Secondly the call for insight held back the completion of the multiplication of operations and consequently, the product of variables and transformations, let alone the entirely formal multiplication of meaningless symbols, which only comply with algebraic equivalences. The historical contradiction: object-directed versus relationship-directed, is reflected in education, which is the central point of the third part. The instruction to multiply a with b yields indeed ab, but not that ab = c. This equality is to be considered as either an object or as an activity, consequently it is to be reduced to intuition or to algorithmical handling, according to the stream to which one adheres. The comparison of objects occurs with configurations or models that belong to repeated addition, ordered pairs, area and size changes. The equality as an activity amounts to the application of a theory, in short an algorithm that leads from the left to the right side. In the structuralistical view the comparison of objects leads to misconceptions. This is easy to demonstrate with the construction rule $1 \times a = a$, which has no figurative representation.

1. INTRODUCTION.

"Say that a boy has a great fladity and that he can maintain it for a little while", what can you conclude from that? Not much I think. Let me express the phrase in a more quantative manner. "Say that a boy has a great fladity, and he can maintain it for three hours", what now? Still no idea! "Say that a boy has a fladity of four, and that he can maintain it for three hours". Many would assume fladities of something like seven or twelve because those are numbers with which you can operate.

Fladity is of course an invention to enable me to show that a term does not necessarily lead to mathematical operations. Originally velocity was used instead of fladity, so "Say that a boy has a velocity of four, and he can maintain it for three hours." Can a process be expected now? Yes, though not without interpretation. Presumably the boy is walking and then there is sufficient information in order to reason: the boy is walking at four kilometres an hour, that is an acceptable velocity for walking. Four kilometres are walked every hour and so the total distance walked by the boy is 4km + 4km + 4km = 12km. The crux is the knowledge that velocity is distance per unit of time and the presumption is that it is in kilometres per hour. Nothing prevents us from thinking that the boy is an astronaut with the velocity of Mach 4. Both interpretations are realistic though, no one walks or flies with a precise velocity. There are continuing fluctuations. The distance covered depends on time and is only linear by approximation. Let us not digress[0]. My question is: apart from the definition (v=s/t), how can it be made plausible by intuition that indeed this distance can be expressed as a product, namely the velocity times the time (s=v.t) ?



[0] Then the distance is the determined integral of the velocity and does not alter the demonstration because the limit of $\sum$vdt is taken.

This is all motivated by a passage from a book by Henri van Praag (p. 156): "Say, that I walk at 4km per hour, and that I maintain this for 3 hours, then the distance covered by me is s=v.t, ie. 4×3 = 12km". Is this a clear story? The answer depends on the person addressed, in any case it is a mathematically correct story. He formalized correctly by applying the right formula to the situation, which originates from v=s/t. Someone with little mathematical background would not recognise the product situation and would perhaps reason: the distance covered is the sum 4km + 4km + 4km = 12km. However Van Praag applied the product. He sensed the difficulty and followed with: "You can make this clear by simply drawing a rectangle with the velocity on one side (or axis), and time on the other." This explanation cannot be much clearer since the observation does not function for the formula s=v.t. The figure proves nothing. As an aid the figure leads to confusing associations. Until the fourteenth century people would not have understood much of this explanation, especially such a grid. The issue is that a distance is compared with an area, consequently length is an area. What is required in this case is a functional understanding of the product and that did not develop until later. In the next section you will find the development of the product in relation to the intuition.

2. AN HISTORICAL SKETCH.

In the Greek era the relationship between velocity and distance caused a problem for many. I am thinking of Zeno's paradox of Achilles and the tortoise. Achilles could run very fast and a tortoise as everyone knows is very slow. Even if the tortoise were given a head start Achilles would quickly overtake him. How do we know that? Experience gives the answer, it can be clearly seen. Suppose that a blind man heard this and said: no, that is not true, Achilles would not pass the tortoise because the tortoise would not be standing still, therefore time and again as Achilles makes up the head start the tortoise has a new, though smaller head start. Zeno, the student of Parmenides, said that the reasoning of the blind man was not correct and every one will agree with that. What is actually incorrect? According to Zeno the fault lay in that the blind man did not see, sorry he did not take the distance as one but he had divided it up into pieces. In short, a multitude does not exist and so Zeno proved to be a fine student, for it was Parmenides who said that the All, which is everything that exists, is one. In this way Zeno used the paradox in order to solve the part/whole question. Actually, that issue refers to the ideas, so is an idea a unit or is it a multitude of ideas[0] for this Zeno is not really convincing. Plato already remarked that Zeno stayed to close to the sensory level. "People try through the practice of the dialogue, without the intervention of the senses, exclusively through reasoning, to reach the essentiality of every thing..."[0]. Dialectics is therefore reasoning namely with the ideal things and activities, the illustrative level is purely an aid to achieve reasoning. Plato kept mathematics and the physical world strictly apart. In consequence the intuition played no role in mathematical evidence, which was to be exclusively a thinking process. Which thought mistakes did Zeno make? Actually the same ones as the blind man, both did not regard the situation mathematically: they identified velocity with distances and added them up, when it was really about the product of two different quantities: velocity and time. After some time v.t is much greater for Achilles than for the tortoise. Here observation produces the wrong reasoning[0]. The product appears not to be the addition of 'visible' values. The paradox is founded upon the interweaving of seeing and thinking. It is an indication of naivety that people pass over the intuitions in mathematics. They can work just as well as a catalyser or a break and for education it is essential to be well aware of both characteristics.

Arithmetics was of importance already with the Babylonians. From clay tablets we know that the ancient Sumerians could multiply as early as around 2500BC. The solution of a question such as: If you take the side off a square then the result is 14.30, which we would write as $x^2 - x = 14\ 1/2$, was already seen on a clay tablet from around 1700 BC. The arithmetics were more operational than structural, people mostly looked for algorithms to solve other problems.

Research into the properties of whole numbers, that is the structure or number theory, began with the Greeks. They sought the perfect numbers, where the sum of the separate divisors is the same as the number itself, for example 28, which has the divisors 1, 2, 4, 7 and 14. The number was

[0] As Socrates remarked, see Plato's dialogue 'Parmenides' 129.
[0] Plato, 'The Republic', 532a.
[0] For a more extensive mathematical elaboration, see Spallek.

fundamental, in their opinion all knowledge was founded in the relationship between numbers. In music they found that the base tones behave like certain whole numbers and the elements fire, air and water existed in the ratio 1:1:2 etc. For the Greeks geometry stood at even a higher level because the structure, ie. the theory of its characteristics, was further elaborated. An important problem was the quadrature of a figure, ie. a square constructed with the same area as the given figure. From philosophical roots, their numbers were solely the natural numbers. Fractions were the relationships between natural numbers. They knew that with segments of a line irrationality occurred and so consciously abandoned expressing length in terms of numbers. Therefore their theory of areas comprised for the most part of the comparison of areas by splitting the one figure into parts so that put together in another way it created the other figure. Their geometrical product understanding was objectively illustrated by the area as the sum of parts. Nowadays our product, also the geometrical one, is a functional connection and therefore a relation. In this, we represent length with a number as we do the area of a triangle (base × height) which is a function of two numbers. Euclides did not reason in the same way and kept numbers and values carefully apart. Because length was not represented by a number or letter x^3+x^2+x was useless in that era, on top of that contents, area and length could not be added together.

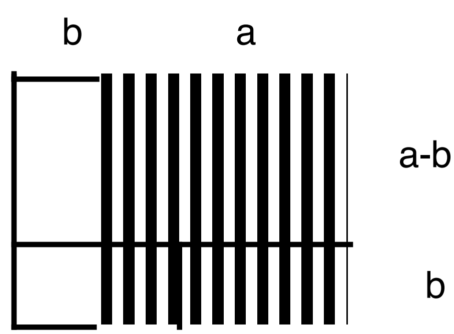
The historical development of mathematics cannot be understood without considering the great influence of Greek philosophical thinking. My theme now is the intuition of the product and so I shall limit the explanation of Greek thinking to three aspects:

1. The object changes from real thing to idea,
2. No self-creation of objects for example by inverse operations, and
3. The lack of interest for the dynamic, for the changes as objects.

The first point is fundamental. The structure of reality is the same as the structure of the world of ideas, the latter is indeed not visible but everyone has direct access to it through anamnesis. Consequently we know the world of mathematical thinking. The second point is of great importance because it turns out so negatively for mathematical thinking. The product is actually a mapping: $(a,b) \rightarrow c$

The mathematician asks himself for example if every element c is a product of equal elements, ie. is there one a so that $(a,a) \rightarrow c$? Can every c be decomposed into equal factors? Greek philosophy says no, the present mathematics says yes, but it has taken two millennia to put aside the no. Now we create objects from the assumption that a product exists, therefore $2 = (\sqrt{2})2$; $2 = 3. 2/3$; $-2 = 2.-1$; $-1 = i^2$.

The last two equations are especially difficult because without extra definitions, such as directed areas, they cannot be observed as rectangles. Focusing on the dimensions slowed the development of the product ab. In the Greek era ab was considered arithmetical, as a product of numbers, or geometrical, as the area of a rectangle and in a statical sense. The a and b were not variables of number types, ab was an operation described by examples. The algorithms developed from that were expressed by configurations. Therefore not $(a-b)(a+b) = a^2-b^2$ but:

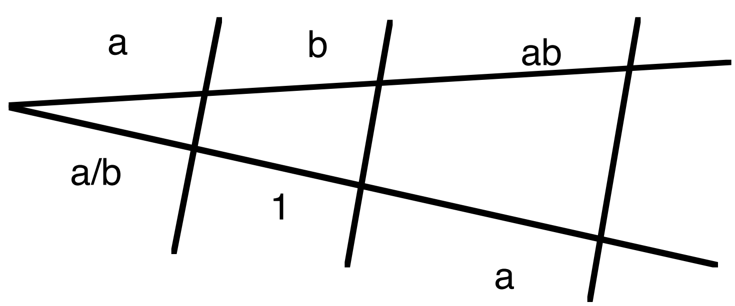


It instigated mathematical algebra whose mathematical figures were used for algebraic comparison. In modern notation ab is the algorithm as well as the result, but in Greek mathematical algebra it is only a figurative operation.

The third point, the static orientation, was so deeply rooted that algebra as a science of general algorithms could not develop until the seventeenth century by conceiving ab as functional, and so as a product of variables. Notice that variables of numbers were not considered as changes, without a number base, until the nineteenth century. The transformation as fundamental concept lead to the development of the group, the system with only one operation. In short, people regarded ab first as a product of numbers or values, then as number variables and finally as transformations.

How did these developments begin? Understandably it began there where thinking in relationships was the furthest elaborated, namely with proportions. The ratio of products was understood as the ratio between rectangles. In this way the product was used in proportions[0] like $BD:BZ = AD^2:EZ^2$ where the ratio between line segments is the same as the ratio between areas. The apriori, the formality in this is geometric, ie. the splitting of a line segments into equal parts and a rectangle into equal squares.

The proportion is a function of an ordered pair of numbers. Every pronouncement about proportions regards four numbers. Understanding proportions by visualising leads to the fundamental difference between internal and external proportions. Antique geometric tradition only allowed internal proportions. In the example BD:BZ = AD2:EZ2 to the left of the equal sign are only lengths and to the right areas. That means that the Greeks allowed only single relationships within one homogenous system of quantities. The uniform motion is characterized by $s_1:s_2 = t_1:t_2$. Changes of middle terms leads to $s_1:t_1 = s_2:t_2$. This is now an equality of two proportions of distance and time and therefore external proportions between two systems. The change rests upon the commutativity of the multiplication and is indeed epistemologically and didactically a big leap and explains the crux of the Achilles-tortoise paradox.



For the rest, Greek insight should not be put down, on the contrary, even with real numbers they were working in a way that was only later matched

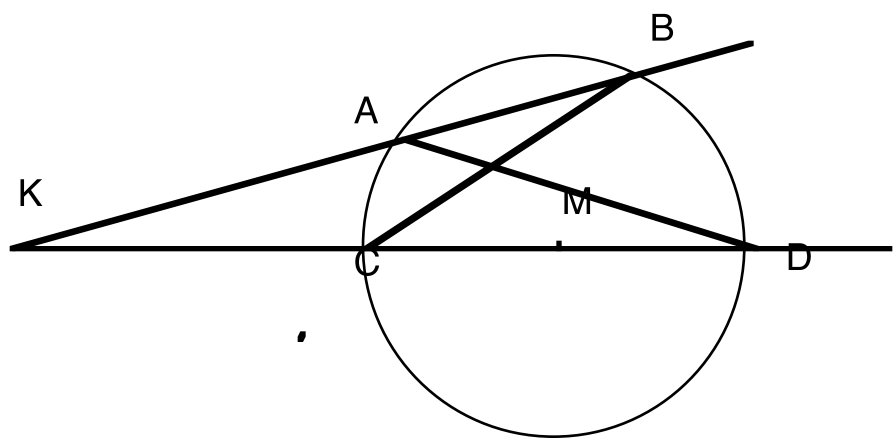
[0] See for instance Archimedes: 'The quadrature of the parabola'. In this piece he is looking for a rectilinearly confined area which, for as far as area is concerned, is the same as a parabolic segment, after it has been proven earlier that this is possible for a segment of a circle and for a segment of an ellipse.

by Dedekind. Eudoxus already introduced the segment, ie. if p,q,r and s are four lengths then $p/q = r/s$ if there are no whole numbers m and n so that $p/q < m/n < r/s$ and $p/q > m/n > r/s$. This state of development was unchanged for a long time. Not until the seventeenth century did Descartes take an important step forward by taking the relationship, here of proportions, as foundation. With this he did not release the geometrical context, on the contrary, Descartes regarded intuition as the absolute foundation of mathematical thinking and deduction only as an aid to explain something[0]. Up to that moment $a:b = c:d$ are two proportions of elements of the same order. The proportion $1:a = 1\times b : a\times b$, as said before was already known in the Greek era, but Descartes posed $1\times b = b$ and that lowered the dimension. Now $1:a = b:ab$ implies that ab must have the dimension of length. Using proportions the product ab retains the dimension of a and b. The consternation brought by $b\times 1 = b$ was great. Intuitive thinking did not allow a rectangle to be the same as a line and when later Cantor also proved that rectangles and lines contained the same number of points and more generally the 1-1 correspondence of all Euclidean spaces, all signals went to red. How can a part contain as many points as the enveloping whole? The situation became worse when Peano constructed continuous images which raised the dimension.

Another problem was multiplication and negative line segments within geometry. Carnot (1753-1823) wrestled with this. Two lines through the point K cut a given circle at A & B and C & D respectively. AD and BC are antiparallel so that the sides of the triangle KAD and KCB are proportional and therefore $KA\times KB = KC\times KD$ (*). The proof does not change if K is in or out of the circle. Therefore Carnot considered the situation correlatively, that is he let K move continuously from outside to inside the circle. The figure is then no longer static. Something essential has changed, ie. $AB = KB-KA$ for K outside the circle and $AB = KB+KA$ for K inside the circle. As the change of K is taken as continuous then KA changes sign. What precisely is happening? How can you represent a negative line segment? Carnot researched this problem further by contemplating line bundles through K outside the circle (M,r). Say that the length between the two points of dissection is 2p. Can you

[0] For example, if all A have the quality B and $a \in A$, then a also has the quality B. Descartes regards this conclusion as superfluous for it is already contained in the major.

construct the line l_{2p} with every $0 \leq 2p \leq 2r$. This is an existentional question. Therefore $KM = a + r$ where $a \geq 0$ and $r > 0$. Construction is meant in the Euclidean meaning, merely to be used for circles and straight lines. In fact it is a question of continuity. Is every in between value reached?



If CD goes through M and KA = x then the equation (*) becomes:

$x(x+2p) = a(a+2r)$

$x^2 + 2px = a^2 + 2ar$

$x^2 + 2px + p^2 = p^2 + a^2 + 2ar$

$(x + p)^2 = p^2 + a^2 + 2ar$

Except for the cross multiplications every calculation has a geometrical interpretation namely as the sum of *areas* of rectangles and squares.

Circle (K,x) cuts the (M,r) in two places and connecting them with K produces the required line.

$(x + p)^2 = p^2 + a^2 + 2ar$

$x + p = \pm \sqrt{(p^2 + a^2 + 2ar)}$

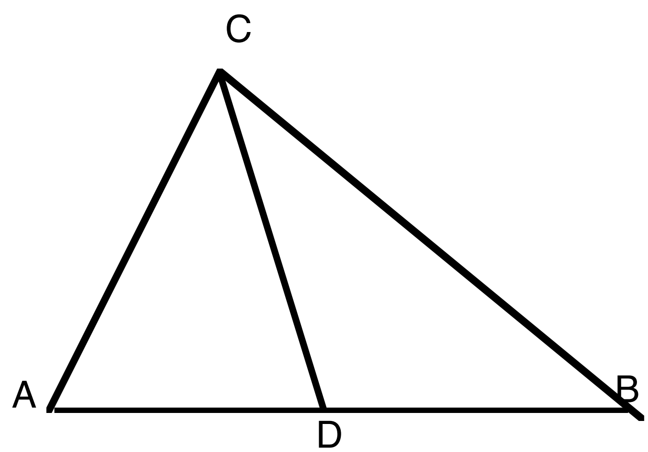
$x_1 = -p + \sqrt{(p^2 + a^2 + 2ar)}$ and $x_2 = -p - \sqrt{(p^2 + a^2 + 2ar)}$

The discriminator $p^2+a^2+2ar = p^2+(a + r)^2- r^2$ is likely to be positive because $a > 0$. Therefore there are *two* values which comply. But x_2 is *negative*. How can KA be positive as well as negative? This question intrigued many mathematicians in the eighteenth century. The problem is that all operations are intuitive except for the square root. Carnot thought of a solution: say that KB = x in the equation (*): $(x-2p)x = a^2+2ar$, which also has two roots x_3 and x_4, where $x_3 = -x_2$ and $x_4 = -x_1$.

Carnot introduced two archetypes that continuously change from one to the other. 'Archetype I' is the figure with $a \leq x \leq \sqrt{(a2+2ar)}$ and 'Archetype II' is the figure with $\sqrt{(a2+2ar)} \leq x \leq a + 2r$. The pointless x_2 in archetype I has meaning in archetype II. In the abstract intuition of both archetypes simultaneously all the roots x_1, x_2, x_3, x_4 have meaning.

The lesson to be learned from the problem is that even in geometry, where algebra is used, algorithms, in this case the conversion of a proportion into a cross-product, cannot be illustrated in a configuration. The theory only works globally and not locally.

Eventually the Platonian conception prevailed therefore intuition was only an aid to the mathematical thought process and sometimes the figurative representations were more limiting than useful. The figurative world is not objective reality for mathematics. Formalising is not always coupled to figurative imagination. Mathematical activities produce their own objects. Decisive was the discovery of algebraic numbers which, as opposed to whole numbers, cannot be unequivocally decomposed into factors. Hancock (introduction vi): "This very perplexing condition was later overcome in part by Kummer's discovery of ideal numbers, which, although of a somewhat fictitious nature, are *not* 'mere fictions'. As Kummer would express it, they are like certain chemical compounds which have their reality in their combinations." By decomposing -1 into i^2 a new element arises, as -1 arose from the additional decomposition of 0 in +1 and -1. The operation is so to speak universal, in general the mathematical activity is prominent. The objects necessary for the intuition are not a condition for the mathematical activity but a consequence. When the one form is the algebraic equivalent of the other then the meaning of the symbols is not important. This liberating notion, which had arisen earlier but only taken shape at the beginning of the nineteenth century, was the base for the following gigantic developments in algebra. First there was the de-counting of numbers as soon as for instance the irrational number and variables were accepted, then the 'de arithmetization' of algebra by the transformation in order to arrive at the algebra of symbols. The existential activity, as we know today, brings forth the mathematical knowledge. Consequently the opinion broke through that the term ab in 1:a = b:ab should be taken formally, as the symbol of an activity and not as an object expressing a rectangle. Likewise $b \times 1 = b$ is the formal *algebraic* and is just as determining for the product as the associativity $a(bc) = (ab)c$. Formally also $a^2 \times 1 = a \times a$ and written as a proportion $1:a = a:a^2$ the a^2 has no illustrative meaning, pure formalistic thinking is required. In mathematics the real number 'a' has the line segment with length 'a' as its object or representation. This line segment can be observed and we regard it to be covered with 'a' units. Say that a^2 is also represented by a line segment with a^2 units. Choose on the angle A the points B and C so that $AB = a^2$ and $AC = a$ and D is on AB so that $AD = 1$.



From the formal $a.a = 1.a^2$ you can decide that $1:a = a:a^2$. Two sides are proportional and the same enclosed angle, so is $\Delta ADC \cong \Delta ACB$ and consequently $\angle C = \angle D$ and CD and CB are antiparallel. The formal $1:a = a:a^2$ mathematically means two homeothetic triangles or the anti-parallel pendant.

Realists understand through representation. The proportion $1:a = a:a^2$ is to be understood by visualisation. Mathematicians take the formal $a^2 \times 1 = a \times a$. The idea that the symbol a^2 can be a number and a length goes too far for students who have not separated themselves from intuitive thinking. The length of line segments refers to objects and not activities. The object changes are a primary source of misunderstanding, therefore in the following chapter I shall explore more deeply into the nature of objects related to the product.

3. EXPERIENCE ORIENTATED VERSUS STRUCTURE ORIENTATED EDUCATION.

How do you get from reality to ideals, ie. from objective space to ideal space and vice versa? From a long time ago this bridging has been an intriguing philosophical and didactical theme. The two antagonistical answers are: with the spirit or with the senses. Further more there are all sorts of in between positions based on the complementary stand point: all knowledge comes about by the cooperation of intuition and thinking. Therefore the importance of intuition is determined by how well thinking functions and forms ideas. Intuition is the activity which focuses attention to something which is an immediate fact. The intuition precedes thinking. This aphorism of Kants is well known: "Anschauungen ohne Begriffe sind blind, Begriffe ohne Anschauung sind leer" (Intuitions without concepts are blind, concepts without intuitions are empty). In fact, Kants conception blurs the dividing line between mathematics and experience, which had been so strongly drawn in his time. The fading began by linking mathematics to pure intuition which can be reduced to geometrical figures. On top of that the fading was strengthened by the notion that all mathematical understanding is constructible. Constructibility together with intuitivity easily leads to the image that mathematical concepts are realisable. For education the necessity of intuitivity is very attractive. Just as for empirical subjects, it is possible to use illustrative education for mathematics, where the students receive the discussed matter physically or in images. The possibility of illustrative mathematical education depends in principle on the empirical or idealistical position and is therefore a high ranking philosophical point of controversy. However, does mathematics allow a choice, although Aristotle contended: "There is nothing in the spirit that is not first detected by the senses."? The crux is that the relationships between real things are of a mental order which therefore are not visible.

My paradigm is multiplication. The theme is therefore narrowed to: "*Is multiplying an object and so intuitive or is it the form which shapes human experience?*" Kant also sought what shapes human experience: "das Formale an der Erfahrung", just as many Greeks who already operated the rule that the general cannot be concluded from the special, and therefore not the idea from the idea nor the ideal from the real. In other words, the general precedes the special. The form on experience is the result of a quest. An opinion about A is analytical if it is contained in A, otherwise it is synthetic. The form on experience contains the synthetic opinions apriori, since falling back on experience, the synthetic connection of the intuitions, is by definition not possible. Experience is of no assistance to mathematics, therefore mathematical propositions are apriori propositions and not empirical. Nevertheless Kant said that a mathematical proposition is synthetic since *the intuition is always required*[0]. Summarizing, there are essentially three different positions:

- The form in the experience. Leibniz spoke of the 'l'harmonie préétablie'. The form and experience are the two faces of the same object.
- Kant's view of the form on experience, ie. the form must be able to be expressed in that experience.
- The form on experience, but it is not based on it. In the words of Poincaré the form is a convention. Experience is suitable for distinguishing which convention is the simplest.

The issue is that the task of multiplying a and b produces ab but not that ab = c. Must one seek outside the concepts and use intuition for c? The multiplication ab is an activity and c is an object. With what activity are we concerned here? The present mathematicians see every equality $x = y$ conventionally, ie. the same as the activity $x \Rightarrow y$, by a calculation or evidence based on algorithms. The conventions are in principle arbitrary but in practice the scientist follows methodical rules. One of the rules is localizing, ie. the division of the global into smaller parts, in other words reducing the compound to the single. This methodical rule defines the product $E \times F$ of two sets E and F. The set is the total of elements and is governed by the concept total and the concept element and nothing else. The localizing means consider the product ef with $e \in E$ and $f \in F$. Usually (e,f) is written for ef and that is the end of

[0] Kant illustrates this important pronouncement with the arithmetical equality 7 + 5 = 12. According to him, this addition is synthetic, because 12 is added, and it is an opinion based on experience. For, he says, "I first take the number 7 and for the notion of 5 I take the fingers of my hand as illustration. Then I add the units which I previously put together to make up the number 5 one by one to the number 7 and like that I see the number 12 arise". See 'Kritik der R.V.B' 15.

the determination. The product E×F is the total of the (e,f) and is as such a new set.

The multiplication of natural numbers is also based on localisation, on the division of b into ones followed by another operation, the addition. The sum of natural numbers is calculable by following formal rules. The required activity is a formal action. If a+1 := a′, the count successor of a and if the associative law applies a+(b+c) = (a+b)+c, especially a+(1+c) = a′+c, then the algorithm gives: 7+5 = 7+(1+4) = 8+4 = 8+(1+3) = 9+3 = 9+(1+2) = 10+2 = 10+(1+1) = 11+1 = 12. Experience is not necessary, it is sufficient to apply the structure rules. In a+1, and therefore the number successor a′ lies the essence of the addition. It is easy to see that the reverse is also true and so each b can be split as the sum of ones. The distributive law a(b+c) = ab+ac localises every product ab to a×1. The construction rule: a×1 := a is a convention that repeatedly applied leads to the conclusion c. According to that: 3×4 = 3×(3+1) = 3×3+3×1 = 3×(2+1)+3 = 3×2+3+3 = 3×1+3×1+3+3 = 3+3+3+3 = 12. For fractions the association a(bc) = (ab)c is added. Then (p/q)r = (pr)/q applies because q((p/q)r) = (q(p/q))r = pr. Once more the rules determine the product and not the experience.

In short *ab=c is a theory*, better ab ⇒ theory ⇒ c, an entirety of rules. 41 × 36 is a task where people do not intuitively know the answer and algorithms are necessary. Therefore 41 ×36 ⇒ algorithm ⇒ 1476.

If e and f are elements from the modules E and F then what is ef? It seems obvious to include linearity, which governs the module so much, in the theory to be drawn up determining the result. Start with E×F and take from that all formal combinations of the elements. Combinations such as $(e_1+e_2,f)-(e_1,f)-(e_2,f)$ and (ne,f)-(e,nf) we define as zero. Linearity is therefore assured and $(e_1+e_2)f = e_1f+e_2f$ applies. Such a product ef we call a tensor product and is symbolised by e⊗f.

Of great importance in the study of structures established by composition laws is the group which is completely determined formally by a theory that says that there is a neutral element a1 = a and every a has a b so that ab = 1 and finally the operation is associative a(bc) = (ab)c. The theory making c from ab must have these three qualities for ab to be a group product.

The last example is about groups mapped onto themselves: G→G. The product of mappings a and b is the mapping ab which must again be determined locally, for every element g of G applies ab(g) = a(b(g)). Say that a is squaring and that b is increasing by 1. Both are operations on real numbers. Now $ab(x) = (x+1)^2$ applies, and therefore c: $c(x) = (x+1)^2$.

The study of such structures and especially the development of the product strides ahead through the application of the structure itself or, outside mathematics, in reality or experience. The ordinary students profit very little from structure knowledge if they cannot apply this knowledge, so if they cannot formalise their experiences. With this there is a *fundamental problem*: How do you know that in a certain real situation multiplication is *applicable*? What is the real meaning of multiplication? Does every theoretical element from ab → c have an intuitive meaning and what representations are there? Does the symbolic form of the theory allow all kinds of representations, from real examples to descriptions in ordinary language; more precisely:

1. realistic, a box of peaches,
2. manipulative, handling real objects,
3. representation in configurations,
4. with language, the wording?

The issue is that mathematics as a structure system differs from the mathematics in the experience. Say ΩΩΩ is a strip of three identical blocks. Consider the first pair of blocks which can be seen in two different ways: as part of a whole and in the ratio part to whole. The first is an object, the second a relationship. The object 2/3 and the proportion 2:3 are representations of this. Mathematically seen they are alike. However for 2/3 2:3 must be known and vice versa. In this way, 2/3 is the formalisation of the relationship 2:3 and therefore it is an application. The two third part of an object does not exist independently from the relation of quantities[0]. In general object and activity are inseparably linked. The intuition does demonstrate a fundamental difference. The object is in principle intuitively observable, but not the activity, at least as we understand the symbolic ties to be unobservable.

The assumption of different mathematics inevitably leads to the apriori question, which has precedence? The thought that '*experience creates multiplication*', and so avoiding a definition, is attractive. Multiplication is

[0] Freudenthal (1978 p. 293) even distinguishes between abstract and concrete numbers. With the internal ratios he speaks of abstract numbers and with the external ratios of concrete numbers.

then discernable and not algorithmic. From experience ab = c appears to be an equality and not a difference. If ab = c then ab is an object as well as an activity. As an object it deals with equality, the truism of the contention; as an activity it concerns the difference and so we have a paradox, says Whitehead. The difference usually has the emphasis to make thought superfluous, which is accomplished through mechanizing. In this manner founding and developing, consciousness and activity become separated. The connection is left to the evolution of special forms of appearance. The essence of an object lies solely in the evolution. In experience things and substances possess an irreducible status, the idea is that the relations also have this intuitive status. Methodologically the question arises: 'What is more important, formalising or the analysis of the meaning?' Many think that issues can be solved solely by independent, creative thinking, so without formalizing. Others distinguish a principle difference between the form and reality. The formal product can never be abstracted from certain objects, but is functionally related to reality. The product can be created freely but it is bound by applicability either directly in reality or in mathematics itself. All knowledge is formal for as far as it eventually shapes reality[0]. The form is not reality but a particular perspective of it. 'Particular' because there are more possibilities. So is a number on the one side a counting number, based on the intuition of iteration, and on the other side a measuring number, based on the intuition of the quantity. The difference appears, according to Otte, with the intended application, as the relationship with reality is observed. The different appearances can also be found with multiplications as:

1. repeated addition,
2. ordered pairs,
3. area,
4. size changes.

This is respectively about the intuitions of the joining in real and geometrical situations and the part/whole intuition in geometrical and real situations. The more reality orientated the more phenomenons and vice versa, the more

[0] In the words of M. Otte (1993, p.35): "All knowledge is formal as it arises by giving definite form to some reality".

structure orientated the less models. For example Davydov suffices with one model of unit changes; the unity/multiplicity conception.[0]

The basic idea is that the theory develops itself with application. That seems like a paradoxical assertion because for an application the theory itself must already be given. Further analysis shows that the paradox is only apparent. Many education specialist have separated themselves from this dogmatic education. Their analysis of knowledge is directed at obtaining knowledge with the question: 'How does the student construct a given mathematical concept?' In their opinion the student makes the concept himself. The question already characterises the tendencies advocated. The research has resulted in many schemes for the development of knowledge, which examined closely are nearly all identical, for based on the operative principle. They describe the process from experience to structure, beginning with the observation of things, the handling of them according to instructions, the interiorisation of them through abstracting and their formalisation. Three elaborations will suffice:

1. A. Sfard: Interiorisation → condensation → concretisation.
2. N. Herscovics, J. Bergeron: Intuitive → procedural → abstraction → formalising.
3. P. Gal'perin: Orientation, material handling, verbalising, inner speech, expressing and realistic objectivising.

These are all handling processes of real or represented things, that are gradually being formalised. The crux is the instruction of the action, how does that happen? The answer is clear: Through rules in whatever form, which should be applied to concrete things. This is actually the essence of the application of a theory. The theory is a mental activity that appears in two forms: in rules of operation and in the relationships between symbols. Out of convenience people forget that the students do not invent the rules of operation themselves. This is most radically elaborated in Soviet psychology where the orientation embraces everything that the student needs to understand in the exercise to be made. The orientation is a complete operation algorithm, that the student simply follows in the given sequence. It is radical because of the rejection of the 'trial and error' method, which others

[0] V. Davydov, for further elaboration, see also Nelissen and the criticisms of A. Treffers and by L. Streefland.

wrap up in the intuitive phase of creative thinking. In all cases the implicitly or explicitly given operation plan means, that the student constructing the concept himself is only partly true.

In short, apart from the mixed forms, there are two directions in mathematical education: structuralistic vs phenomenologistic. The structuralists base themselves upon structure analysis and search for concrete embodiments of algorithms for application. The phenomenologists begin with the analysis of the phenomena and look for real and therefore informal admission for the practice of mathematics by everybody. The phenomenologists are problem solvers and aim especially to understand the reality and the development of mathematics for as far as it is visible in reality. Mathematics as a system remains in the background (Treffers, p. 84). It is not too bad if multiplication as a definite algorithm, or more generally as a theory, is not achieved. The structuralist can also begin with a problem of experience which leads directly to mathematical structure via apprehension. In fact the structuralists are system builders. The phenomenologists aim especially at the particular which characterises each problem and demands their creativity for a solution. The structuralists aim at the general which, as ancient Greeks already thought, precedes the particular, from the basic idea of the interwovenness of the concrete and the mental operations. Schematically:

particular (paradigm) → general mathematical theory → application.

The first arrow symbolises apprehension, the second decomprehension.
The phenomenologists work in precisely the opposite way:

initial problem → general method of solution, also support.

The two antagonistic opinions present two entrances for learning. The one, the algorithmic approach begins with the rules of derivation, the other, the conceptual approach works from back to front, and so multiplication is repeated addition. The conceptual approach has the eminent difficulty that children do not get to the construction rules and will never understand what multiplication really is, which appears directly when working with inverse processes like fractions or negative numbers. Mathematising is really the application of algorithms, ie. formalising. The main cause of the impossibility of intuition is in the invisibility of the algorithm, apart from the symbolic representation. The algorithm cannot be captured in one figure. The application of a mathematical theory, however small, does not coincide with figurative illustratability.

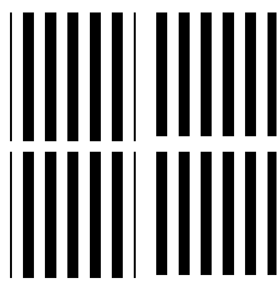
Some teachers will nevertheless avoid 'uncomprehended' formalisms and in fact make it very difficult for their students by trying to empirically support every next thought step, which leads to many misconceptions. They arise when going too far in trying to use the visual illustrations of mathematical activities. A poignant example is the visualisation of the division 7 : 1/2 using the figure that comprises seven equal boxes in a line and below them fourteen circles. This figure suggests addition and not division. The didactical use of metaphors is limited. Somewhere there has to be a major change to fundamental methodic legality. If this does not happen then everything will be reduced to addition, also division and with that the closely associated equality, linearity and linear mappings. Consequently division will not be under discussion, only distribution. Therefore not the inverse operation of multiplication but of addition, ie. of metaphors like 'combining' and 'together'. Division of dimension numbers does not exist, but relationships to them do, and so there are two different perspectives: combining or omitting and ratio. In the learning process both activities must then be abstracted into dividing. However they are manipulated one is always faced with a didactical crux, since mathematical thinking clashes with abstracting. Mathematical thinking is applying the general to the particular and not the other way around. Just like physical induction clashes with mathematical induction. In mathematical thinking an element performs like a general element, but a real presentation cannot. With a visual presentation division is introduced as repeated substraction. 7:2 is not a problem, repeatedly take two things together away from seven things. What do you do with 2:7? The method is to repeatedly remove seven things from two things. How can 7 be repeatedly removed from 2? If the student is trained in them, he will think to work with deficiencies, and then he will be completely lost in the dark wood of intuitions.

The dogmatic and heuristic methods are opposites, but that does not mean that the dogmatic method does not use figures or real material, they only have different functions. The dogmaticist does realise that the relationships constitute the objects of mathematics, that is why the figures, or more general materialising through visualising, have to meet the following requirements. The figures must:

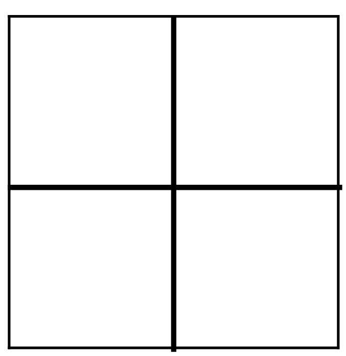
1. Call up relationships and rules through associative and metaphorical thinking;
2. Offer theoretical overviews, ie. the rules in totality;

3. Be sufficiently abstract, otherwise the material and the story are remembered better than relationships.[0]

The cause of many misunderstandings lies in the desire for illustrative education which can only direct itself to the particular and therefore not to the general, which constitutes the object of mathematics. In the intuition there are *levels* to be distinguished:



1. The real observation, ie. the ordinary perception, which is directed at the special and in its phenomena constitutes the basis of mathematics. The example is the representation of four tiles, which are simple to count.



2. The intuition of the abstract, the perception of abstract objects in figurative illustrations. For example the representation of squares in the same configuration as the tiles. For the intuition of the abstract, foreknowledge is necessary; here knowledge about squares. It then appears that there are five squares and not four. The evidence is constructive, you count the separate squares, possibly drawn separately.

[0] For the relation with Kants notions see Jahnke.

Another example is the representation of the product as an area, where the equivalent sets must be seen and counted. ΩΩ together with ΩΩ as an image of 2×2. It is actually an illustration of the distributive law. Characteristic of this form of intuitivity is that the evidence is the construction itself, added to the foreknowledge.

3. The abstract intuition, which is directed at the general, ie. considering simultaneously *all* rules or theory, which determines the inner connections. Consequently this is the 'seeing' of the relationships which are in fact invisible. The figures which are used here are only assistants and not evidence. The reasoning is not based on construction but on what exists and so on its own theory. Ω can now be used to illustrate $2\times2=4$. This is not about the area any more but about the illustration of the commutativity, associativity and distributivity. Associativity does not work here, because it requires a spatial figure. However, for the multiplication of natural numbers associativity is not necessary. The abstract intuition is complete as far as the underlying theory is complete.

Mathematics has freed itself through a long historical process from intuition as evidence and consequently from the non-corresponding intuitions. Take the addition of numbers which like every operation is determined by a definition and characteristics for the construction, the lying down of object and activity, so to speak. Determining the object is performed by $a + 1 := a'$, which is an expression that stands above intuition. Phenomenologically considered to the left there is a measure number and to the right a counting number and they are essentially different also regarding the dimension. The counting and measuring intuitions are equated here. In the representation 'a' is a line segment which lengthened with the line segment 1, gives line segment $a + 1$. Here $a + 1$ functions as a size number, with the line segment as illustrated object, while a' is a counting number in the symbolised activity. Therefore $6'$ is only a symbol, namely 7.

The same goes for the product. The appeal to intuitivity leads to equating the different intuitions inherent in the appearances: repeated addition, ordered pairs, area and size change. The essence of multiplication is again in $a \times 1 :=$

a. The product ab is, as it were, self-referring to a × 1.[0] The construction rules are applied to these building blocks. The issue is ab = c as an object (=) or as an activity (⇒). Phenomenologically a × 1 is an activity and a is an object, consequently they are intuitions which cannot be reconciled with one another. Follow this:

1. Repeated addition

   If n is a positive number then a × n = a + a + a + ... + a so long as there are n terms of a on the right hand side. This arrangement only works as long as n is a natural number and not 1. Since a × 1 = a is not addition.

2. Pair forming

   a points . . . .

   b points . .

   Connect every point of a with every point of b, in this way ab lines appear. When b = 1 there will be a lines. Therefore a × 1 consists of a lines and in a × 1 = a there are lines on the left and points on the right.

3. Area

   If a is the size of one side and b the size of the other then ab is the area of the rectangle. If b = 1 then a × 1 is the area of a rectangle and a is the length of one side. Therefore a × 1 is the counting of squares and a the number of line segments.

4. Size change

   Start with a value b and take from it a fraction or percentage. ab is the changing of b, say the expansion or contraction of b. But a × 1 = a consequently is a scale factor and that is not a size change of 1.

My demonstration culminates in the conclusion that it is a misconception to regard the appearances as multiplications. In fact they are purely applications of multiplication as a structure, that is to say that only the activity concurs and not the object. In phenomenological mathematics people act as if experience is sufficient. In reality there is continuous defining.

[0] Just like the cell of a plant is identical to the plant it self, because in favourable circumstances the cell will grow into that plant. The plant is its own clone.

Take the product of distances. Situations with distances are often complicated. The example in the introduction where the distance becomes an area is testimony of that. The typical complications are found also in the product of distances, which is often the model for the product. Take three cities A, B and C. From A three different roads lead to B and from B two roads lead to C, symbolically expressed as A ≡ B = C, how many roads lead from A to C? The thinker who's knowledge is based on what can be perceived with the senses tells us that there are five roads from A to C and consequently adds the given roads together. Let us call him the observer. As observers, nearly all students correctly answer five, since there are five roads. The questioner, realizing that his question has not been properly perceived, explains his notion of road. He *defines* a road as a combination of two pieces, consequently a road AC is constructed out of a road AB and a road BC. In the previous sentence the word road was used four times but not all with the same meaning. The road can be perceived both in its real sense or mathematically. In the phenomenological conception road is a junction, a sum of two real roads. In the mathematical sense 'road' is a product of two roads, notice that it is a possibility and therefore not a proper road any more. The real object 'road' has changed into a relation 'road'. It was pointed out earlier that the set product E × F consists of the products (e,f) which are actually purely formal, ie. they have no meaning yet. The question was asked to determine the cardinal number #(E × F). In mathematics as a system it can be derived that #(E × F) = #(E) × #(F), which is in this case directly applicable. The observer does not know this formula and represents the six 'roads' separately now, adds them and ends up with six. The product thinker sees the structural activity and reasons as follows: Choose a road AB then you can form a 'road' AC in two different ways and this produces two different 'roads' AC, even if the first pieces are equal. There are three roads AB and therefore six roads AC. The observer will always speak of the sum this way; the product thinker comes up with the terminology 'the product of roads'. With this the product thinker expresses that the six roads are not counted separately, but that 3×2=6 is used. A product can be applied to the action: executing an activity a number of times, in this case 3 × 2, getting three things twice. Schematised this can be represented by 1♦ n♥ oΩ. The object orientated observer-sum-thinker arrives via this scheme at six, and has counted the six things separately. The product thinker pays attention to the activity and represents that in the structure 3 × 2, a three relation of a two relation. The nature of the numbers depends on the opinion: three and two are cardinal numbers following #{#{1♦}, #{n♥}, #{oΩ}} or three is an ordinal number and two is a cardinal number according to #{1♦}, #{n♥}, #{oΩ}. The matter is that the activity determines the thinking object and not the thing. Consequently the object here is 'getting', ie. the product. More generally it means that in the 'unit-multiple' situation the sum-thinker counts the multiples and the product thinker expresses the activity structure. The sum-thinker adds and the product thinker multiplies and therefore they both have different objects in view. The sum-thinker will soon begin to work more practically, taking multiples together in accordance with known additions such as 2+2+2 or 3+3 and therefore view the situation as a repeated addition, though nevertheless as the application of a product.

The distributive law is characteristic of a process with two operations, such as a ring. Both operations, say + and ×, are linked through this. So multiplication gets broken down to addition and inversely the repeated addition changes into the multiplication, consequently for whole numbers 3×2 = 2+2+2 applies. Multiplication as repeated addition gives the correct result for the ring but it remains a misconception because the activities differ. The misconception becomes apparent with inverse operations, subtraction and division, for the latter we arrive at quotient rings and bodies. The misconception is evident in sets with only one operation, like the group. Consider the symmetry group of order n (from the permutations of 1,2,...,n); or the transformation group of the plane consisting of products of transformations, rotations and reflections; or the representation $x \rightarrow (ax+b)/(cx+d)$ with $ad-bc=1$.

Mathematicians use the illustration as assistance. On the other hand for the phenomenologically orientated teacher, learning is founded and enlightened by illustrated education. The didactic difficulty is of an ontological nature. People want to translate multiplications as real activity into lasting object-like entities. Multiplication is truly a complex mental activity whose crux is the objectivising of the activity so that multiplication can be intuitive. Objectivising into symbols to calculate with is not the aim of phenomenological mathematics, it is about the objectivising in illustrative figures so that the knowledge is reduced to mere reception. A condition for that reception is the reality of the relationships.[0]

What is ab? The algebraic answer is clear. The difficulty arises from the question: can ab be illustrated? As an activity there is no image. The formal calculating activity comes from theory, expressed in the algorithm bound in rules and the real counting activity occurs through schemes like the quadrature of a square or the distance model. An algorithm, the formal operation with symbols, is not considered to be an illustration, because it lacks the full overview. The scheme of a product as pure intellectual knowledge can equally not be visualised. According to Kant there is a transcendental scheme to our thinking that is a condition for perceptiveness and is a rule for the synthesis of the imagination. The pure scheme of the size is the number, of which the image can be found in space and time. However Kant did not sufficiently realise the actual freedom of mathematics, and consequently the independence of experience.

There has been much research (e.g. Dekkers, Fischbein, Fisher, Greer, Graeber, Kouba) done into the consequences of learning product understanding using intuitions or experience. Below I will elaborate on a couple of more or less known misconceptions.

The first is well known and that is: the *multiplication makes bigger*, because with addition the number increases; and in reverse division makes smaller. Much has been written on this so I shall elaborate no further.

The second misconception is even worse and that is that the product is *a part of the whole*. The part of the whole is based on the thought that numbers are only possible through a connection with objects of experience. The numbers can then make interconnections but then according to the parts determined by the entire object[0]. An excellent example[0] of such a miscon-

[0] Leipniz (1646-1716) already denied the reality of relationships. To him all true propositions were analytic. For that matter, all great philosophers regarded knowledge as an activity and not as intuition or illustration.

[0] The idea that intuition refers to a whole has a theological background. The absolute intuitive intellect is exclusively reserved for God, who fully knows the whole in relation to its parts. According to Kant, the intuitive intellect goes from general to particular. The notion of a whole contains the base of the possibilities of its shape and the connection of the parts. Kant used the concept of the intuitive intellect to describe the limits of the human intellect. See Jahnke 1989.

[0] This example is by Freudenthal in 'Willem Bartjes', SLO, jan 1982.

ception is the sixth form pupil who to the question of the type 1/3 of 12 kept answering 8. The cause is the fraction model. 'If you eat a third of a cake divided into 12 pieces there are indeed 8 pieces left and naturally the remainder is the answer to the question'. The cake model is a poor materialisation of the idea 'part-whole' and remains so. The part is 1/3, the whole is 12 and so 1/3 of 12 is translated into the object of an object. The mathematical relationship corresponding with 'of' does not come across properly[0]. The problem is worsened by working with decimals. Students do not recognise multiplication with 0,63 as such because they cannot imagine how you can make 0,63 from a quantity[0]. The numbers that appear in the model of the product are mental objects that are accompanied by the word 'of'. Just as half of something, like half a loaf, and not 1/2 by itself. Half a loaf is experience, how is it possible to discuss half of something? Does the idea 1/2 not precede my tale? At first 'of' is associated with subtraction. Just as with subtraction as with multiplication it is neither about parts nor wholes but about activity. The essence of subtraction is that it has a comparing function and that is insufficiently expressed by the verbs take-off or take-away. The one-sided look at subtraction is classical, ie. only the abstraction of taking away, so taking a part of a whole. Subtraction is relating in a particular way. The object is the difference. "This book is thicker than that one, how many pages do they differ?", is an applicable example of the relationship between two quantities and not between two objects.

The product offers two perspectives: as an activity and as an object with the inherent misconceptions, that are expressed in 'of' and '×', and so on the one side a of b and on the other a×b. For example 1/2 of 1/2 and 1/2 × 1/2. The application, the object orientation with the square as the archetype appears as follows n 1; 1/2 is half of 1, and n 1/4 is half of the half. There is no image of it as an activity ie. 1/2 × 1/2. The misconception comes from the fact that *the product is part of a part of the whole*. The part of a part of a whole

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[0] Julia Matthews (1981) explained to children and adults among other things the next problem: "I took 29 apples an you took 15 and ...". The intended continuation is 'the difference is 14' or 'you took 14 more than I'. Her research showed that only 20% of the eleven year olds and about 60 to 70% of the adults could give the proper continuation.

[0] See Kouba 1989, p. 157.

must act as a metaphor for multiplication. The mathematical relationship is easily identifiable by the double taking away as was earlier remarked of in the example 2:7. In the beginning of this century it was already realised that the difficulty originates from the part-whole identification. Product orientated thinking is relationship orientated thinking and product-perception is not joining. *If eduction is not relationship orientated then misconceptions will arise*.

Now consider the area model for multiplication. Many people choose the rectangle in order to provide their own insight into the exposition of algorithmatising. Instead of algorithmatising a three phase process is followed: first the making of different equivalent sets, then representing the elements with squares and finally taking the strips of squares together. 2×3 consists of the strips ΩΩΩ and ΩΩΩ, which can be placed in a rectangle:

ΩΩΩ
ΩΩΩ

A similar model is a rectangle of points:

. . .
. . .

The crossings model in which representing is not by squares but by points, as the intersections of a horizontal and a vertical lines.

Students easily give up during this three phase process. If the students cannot imagine equivalent sets because of the misconception of the 'of' then they surely cannot take them together. Finally the students see this taking together as a repeated addition and not as a product.

The heuristic method of mathematical education all too easily gets students into trouble by teaching them to understand the concepts incorrectly. The dogmatic method is unavoidable and must receive meaning through continual and suitable applications so that mathematics does not essentially remain uncomprehended. Neither algorithmic nor conceptual approach leads to a real understanding. The only thing which remains is harmonic amalgamation.

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