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3 key-word descriptors

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One major thrust of the mathematics and science education reform movement has been close observation and analysis of students' problem solving behaviors (Duit, 1995). Consequently, a number of researchers have observed that in attempting to solve problems in mathematics and science, many students make errors due to erroneous beliefs, attitudes, and assumptions, as well as due to lack of knowledge, faulty logic and poor problem solving and metacognitive skills (e.g. Schoenfeld, 1995; Tall, 1985; Orton, 1977). All of these aforementioned errors are considered to be misconceptions (Skelly, 1993).

There are many studies from all areas of science that have shown that "alternate frameworks" or misconceptions are consistent to a certain degree among students. These "frameworks" are not simply fuzzy ideas, but deeply rooted convictions with high explanatory power for the students. Quite often these convictions are in sharp contrast to the accepted expert view (Duit, 1995). Although a good deal of misconception research has been done in science, a great deal less has been done in the area of mathematics and only a few existing studies have been done in the area of calculus (Orton, 1978; Tall, 1986; Geuther, 1986, Dubinsky, 1991).

To develop an errors or misconceptions taxonomy for beginning Calculus, the best research in science was integrated with the work of Geuther (1986), Borasi (1994), Colgan (1991) and others.

In science education, the most complete and validated model for classification of students' misconceptions was done by Skelly (1993). Skelly's (1993) coding system and codes representing the principal categories of errors were generated from a very thorough and extensive review of misconceptions in science literature. Her coding system consisted of seven principal categories or sources of errors, each of which was designated by a multiple of ten, so they could have up to ten subcategories. An eighth category was used as a utility category, to classify other phenomena that occurred in the interview process that could not be coded by the taxonomies main categories. Category 80, therefore, in Skelly's schema was not an error category.

Skelly's categories were as follows:

10 Language related misunderstanding and misinterpretation.

20 Deficient prior knowledge on the part of the learner.

30 Overtaxing the short term memory of the learner.

40 Mismatch of cognitive demands of the subject matter with the level of cognitive development of the learner.

50 Choice of mental strategy inappropriate to the subject matter on the part of the learner.

60 Low standard of epistemology on the part of the learner.

70 Experiential misconceptions.
80 Other coded phenomena.

Each of categories 10 through 70 contains from one to nine subcategories categories, which are specific types of errors within the category, which were drawn from the literature as well as Skelly's career classroom experience.

Content validation of Skelly's categories was initially established informally by frequent discussion with three experienced Chemistry teachers as the codes were developed and refined. Final content validation was established by submitting the classification to two other experienced teachers of Chemistry for comment and critique after the codes were finalized at the end of the study.

The version of the Calculus errors categorization taxonomy used for the present study is a modification of Skelly's (1993) taxonomy. Although Skelly's classification and validation was done for errors found in learning by Chemistry students, the categories are well defined, exemplified, generic to any abstract learning content and, most importantly, validated.

Our adapted version of Skelly's taxonomy for beginning Calculus is shown below:

10. Language related misunderstanding and misinterpretation of vocabulary, analogies, symbols, and overall meaning.

11. Error related to using everyday meaning for words which have a context-specific meaning in Calculus.

12. Technical words being defined, (for example, with words from the text) and used without understanding.

13. Using a word with a technical meaning inappropriately when applying to specific situation or phenomenon.

14. Misunderstanding or misuse of a symbol.

An error of this type includes two situations:
a. The student does not know the verbal or conceptual entity which a symbol represents.
b. A student confuses one symbol with another.

15. Overextension of an analogy,

16. General misinterpretation or misunderstanding of the overall statement or question.

17. Inability to relate language to its graphical representation.

18. Incomplete definition, not clearly specified.

19. Inability of a student to articulate his/her thoughts.

20. Misunderstandings related to a deficient prior knowledge base on the part of the learner.

21. Calculus fact or vocabulary word not known to the subject.

22. Calculus fact or vocabulary used incorrectly or incompletely by the subject.

23. Common Calculus calculation procedure misused by student.

A procedure is defined here as a routine series of steps frequently used in Calculus to carry out a typical Calculus problem.

24. <u>Algebraic error: Mechanics</u>.

Incorrect solution of a correctly set-up algebraic equation.

25. <u>Algebraic error: Reasoning</u>.
Failure to set up an algebraic expression correctly.

26. <u>Algebraic fact or definition not known or misused</u>.
This is coded when the algebraic content required for a situation is not known or not recognized by the student.

27. <u>Improper use of a calculator by student</u>.

Insofar as possible, the interview transcripts should include all student comments which indicated a problem with producing correct answers from the calculator. Therefore, the coding of these errors is reasonably straightforward. Working back from students' incorrect answers for a problem which was set up correctly is another method of determining if this code should be used.

Graphing calculators are used in most Calculus courses since the original Skelly classification was developed and validated. These errors made by students‘ improper use of the graphing calculators may differ from those errors student make with misuse of regular calculators.

28. <u>Confusing a newly introduced concept with a previously learned concept.</u>

29. <u>Previously covered material has been forgotten or confused</u>.

30. Error due to overtaxing the short term memory of the learner.

This documentation of errors due to overtaxing the short term memory of the subject is problematic in this research, as it was for Skelly, because the procedures employed are not specifically designed to detect this source of error. However, when analyzing an interview, a knowledgeable Calculus teacher is likely to recognize when a student has not sufficiently processed a Calculus procedure so that the procedure functions as a single unit in short term memory. Thus, overload results. In this study, as in Skelly's, no further breakdown of this category was practical.

31. Misunderstanding caused by too rapid coverage.

40. Error due to a mismatch of the cognitive demands of the subject matter with the cognitive development level of development of the learner.

Although research has established this to be an important source of conceptual errors in learning, it is not expected to be evident, as in Skelly's study, in the interview format used. This is listed as a limitation of our study.

41. Subject matter is too abstract or formal for the developmental level of the student.

50. Error due to choice of mental strategies inappropriate to the subject matter.

51. Use of algorithms without understanding

52. Rote learning of material not yet understood.

53. Error in logic (including getting the right answer for the wrong reason.

60. Errors due low standards of epistemology on the part of the student. That is, errors due to insufficiently rigorous standard of knowledge on the part of the learner.

61. Guessing.

This subcategory is intended to be used to be used to identify straightforward and intentional guessing only

62. <u>Tolerance of illogical statements or conclusions (Learner is generally aware of the error.)</u>

63. <u>Insufficient scrutiny of answer or conclusion (Learner is generally not aware of error, but is expected to be.)</u>

70. Detection of or facilitating of error correction

This category is not an error category, but rather a record keeping category that Skelly used to record occurrences of investigator actions and questions, and student self corrections.

71. <u>Probe questions used to cue or encourage an answer from the subject.</u>

72. <u>Interviewer summarizing, giving correct answer/interpretation.</u>

This procedure was necessary in order to facilitate getting past a concept which the subject clearly did not know. It is essentially a STOP mechanism, by which the interviewer can move past an impasse in the interview process.

73. <u>Student self correction. (Not including simple mis-speaks.</u>

74. <u>Student makes same error after previous correction by interviewer.)</u>

The procedure used in developing the codes for our taxonomy is described in the following section.

<u>Development of the Codes</u>

Our initial macro-classifications representing the major categories of misconceptions were established from the misconceptions in mathematics literature. Our original macro-classification schema consisted of ten principal categories of misconceptions, each of which was designated by T(1-10) after Borasi (Borasi, 1985). These categories were:

T1 Student never knew how to solve the problem.

T2 Student has deficient mastery of prerequisite skills, facts and/or concepts.

T3 Errors due to incorrect association or rigidity of thinking.

T4 Errors due to application of irrelevant rules or strategies.

T5 Errors due to language difficulties.

T6 Student may need more time to complete the problem.

T7 Algebra error.

T8 Errors due to reaching a dead end.

T9 Missing data.

T10 No attempt to solve the problem.

These categories did not contain subcategories.

Subcategories list specific misconceptions made by the student within each category and would be useful in more clearly defining the categories. In a personal conversation with Borasi (May, 1994), we found that Borasi did not have definitions of these categories, nor subcategories which would be helpful when using this taxonomy for error classification.

The second taxonomy we considered in our efforts was a modification of Colgan's (1991) classification schema. This classification schema consisted of eight principal categories:

E1. No answer

E2. Misuse of data

E3. Language error

E4. Logic error

E5. Distorted definition, theorem or rule

E6. Incomplete solution

E7. Technical error

E8. Lack of knowledge

This version was used by Colgan (1991) to classify errors in a college level finite mathematics class. This version had definitions available for the categories but did not appear to have been validated.

Taxonomies

Some of the literature on taxonomies of errors will be discussed here to give the reader some background on the work that has been done in this area in mathematics.

Radatz, (1979)

Radatz (1979) proposes an information-processing classification of errors obtained by examining student errors in completing arithmetic calculations. Radatz's investigation of possible causes of student errors led him to suggest that errors are a result of definite processes whose nature must be discovered. According to Radatz, such an analysis could serve as a point of departure for investigations into the processes by which students learn mathematics.

Vinner, (1981)

Vinner et al proposed another classification scheme which deals with the specific content of addition of fractions within the more general content area of arithmetic. The researchers' initial investigation of incorrect responses resulted in the following list of categories:

1 Wrong reconstruction of detail: Partial forgetting

2 Misidentification: Use of an algorithm which applies elsewhere but is not appropriate for the situation at hand.

3 Wrong analogy type: Generalizing inappropriately.

4 Wrong interpretation of symbols.

5 Compartmentalization: Failure to use existing knowledge to check results in a new context.

Matz, (1982

Matz developed a unifying account for some of the observed systemic errors in high school algebra. She proposed that errors are due to reasonable though unsuccessful attempts to adapt previously acquired knowledge to a new situation. As a result of her initial investigations, Matz (1982) claimed that the extrapolation techniques of linearity and generalization were both used and misused most often.

Geuther (1986)

Geuther's categories are a modification of Vinner et al (1981) and Matz (1982). Geuther did not claim that the list of categories was comprehensive, or even distinct. The categories simply attempted to describe systemic errors (Confrey & Lipton, 1985) made by a sample of first semester calculus students in solving calculus test problems. The main purpose of the categories in Geuther's study was to serve as a focus for hypothesis generation and student questioning techniques to be utilized within a constructive grading format. As such, the categories were not employed in any of the analyses of Geuther's study

.

Borasi, (1985, 1992, 1994)

Borasi designed a study to explore how secondary school students could be enabled to capitalize on the potential of errors to stimulate and support mathematical inquiry.

Borasi's taxonomy of errors was modified and used as version 1 of our taxonomy, which was tested in a pilot study. We were able to classify the student errors using Borasi's taxonomy, but since there were no operational definitions given for these categories, we had to use only her examples from her studies and made our own decisions as to what she may have meant by her categories. This problem was the chief impetus of our shift to Skelly's model.

Movshovitz-Hadar et al., (1987)

The system developed by Movshovitz-Hadar, Zaslavsky, and Inbar (1987) is an empirical one. It is the outcome of a content analysis of students' solutions to two samples of problems on a variety of mathematical topics. They developed a system of six error categories by studying students' written work on a graduation exam in Israel. This category system includes the following six descriptive categories of errors as a model for classifying errors in high school mathematics:

1. misused data
2. misinterpreted language
3. logically invalid inference
4. distorted theorem or definition
5. unverified solution

6. technical error

Each of the six categories are described in operational form and characteristic elements of each category, examples of errors and error analyses are given in detail in Movshovitz-Hadar et al (1987).

Colgan, (1991)

Colgan 1991, followed in the direction that Movshovitz-Hadar, Zaslavsky, and Inbar had suggested and used their classification system to consider if students consistently make the same type of error across several topics throughout a one-semester finite mathematics course. His pilot study and subsequent reliability testing of the classification system showed that a modification of the system developed by Movshovitz-Hadar, Zaslavsky, and Inbar (1987) was effective for classifying errors in finite mathematics.

With our discovery of Colgan's classification system, we found a system with operational definitions, that was reliable and was used to classify college mathematics students errors. It appeared to be much closer to a classification system we could use in our study.

A brief description of Colgan's error classification system is as follows:

A- Misuse of data (e.g., copying errors)
B- Language error (e.g., incorrect set up of word problems)
C- Logic Error
D- Distorted definition, theorem or rule
E- Incomplete solution (e.g., incomplete row reduction)
F- Technical error (e.g., basic skills in arithmetic)
G- Lack of knowledge (e.g., little or no work given).

Although useful and more complete than the other systems just mentioned, Colgan's schema is not as comprehensive, detailed or generic as Skelly's (1993).

Skelly, (1993)

Skelly developed and validated a classification of student errors and misconceptions in chemistry. Because there were so many subcategories and because it was validated for chemistry, we at first believed that it would not be appropriate for our studies. But we kept coming back to it during our consideration of other models and soon realized that it was the most appropriate model for a taxonomy for student errors in calculus in the literature. The more we fit these other models to it, the more we saw that it was indeed generic, well defined and validated and could easily be used for classifying calculus errors.

So basically, our taxonomy integrates the work of Borasi, (1992); Movshovitz-Hadar et al., (1987); Colgan, (1991); Skelly, (1993). Our taxonomy uses Skelly's misconceptions macro category derived for science and uses Borasi and Colgan in the area of mathematics to provide specific items for each of the misconceptions. Initial piloting of the taxonomy was quite positive.

Pilot study

The pilot study data collection began during Spring semester, 1995. The sample consisted of six volunteer Calculus I students at a Massachusetts state community college. During the pilot study, a method was developed and tested for classifying and validating the classification of errors/misconceptions that Calculus I students have regarding the derivative and applications of the derivative.

The interview protocol for the pilot study was developed according to Fowler & Mangione (1990) and others. The interview questions were based on information in the Calculus I text, the solution manual accompanying the text and the test bank developed to accompany the text. The interviews were taped recorded and transcribed verbatim. A copy of students' written notes were kept to possibly clarify, if necessary, the transcriptions of the recorded interview. The students' error/misconceptions were coded and classified using the original taxonomy, generated by the literature review, of errors/ misconceptions. This original taxonomy was a modification of existing taxonomies (Radatz, 1979; Borasi, 1992).

The classification procedure consisted of reading the transcript (written responses), circling each error, checking the appropriate column on the coding sheet. We used this procedure to degree to which the errors could be consistently categorized into the proposed classification. A revised taxonomy of errors/misconceptions was developed using information from this pilot study and a continuing search of the literature.

The present study

The present study is in the process of being conducted. The purpose of this study is to more extensively validate the taxonomy of conceptual errors and misconceptions that we have developed and to describe errors as they occur during concept acquisition in a Calculus I course. The specific concepts addressed in this study will be the derivative and the application of the derivative.

In our current study students attended Calculus class with four lectures of fifty minutes per week for four weeks. The class was be taught by an expert Calculus professor. Students were interviewed on their understanding of the content of the class material using a standardized think-aloud interview protocol that has been developed and field tested. The content of the interviews was based on an analysis of the text book and supplementary materials, such as student's solution manual.

The error and misconception taxonomy that has been developed has been used to code student work-aloud protocols in terms of the types of errors and misconceptions students make. The interviews will be scored by us and another expert calculus professor who we are training in the protocol and scoring procedures. A consensus session will be held to discuss any categories for which there is not a consensus from the group. A sample of 14 public community college students have been interviewed in this study. These subjects were volunteers.

Limitations

Our work has only analyzed errors/misconceptions regarding the derivative and applications of the derivative and our sample was community college Calculus I students who volunteered. These students may differ from other community college students, as well as four year college and university students. Because of the small number of students in the sample, any results may not be generalizable. Certain types of error/misconception sources were not identified in our work. For instance those caused by a mismatch of the student's developmental level and the cognitive demands of the material. However, with all these limitations the results of our work has been very positive and very promising.

Significance

Calculus I is a required course for a large number of students at many colleges and universities (Artigue, 1991). Calculus will also most likely remain the principal point of entry to most mathematically based careers. Many students have difficulties solving the various types of problems presented in the course. Nationally, nearly 50% (Kasten, 1988) of the students who enroll in Calculus I each semester either withdraw or receive a D or F grade. It is clear from these facts that any effort to reasonably understand and improve undergraduate calculus learning is highly significant and important. Lastly, calculus is the area where the least amount of research has occurred in terms of the mathematical reform movement. We hope other researchers will use the calculus misconceptions taxonomy we have presented here to carry out the many studies that are needed in this area, and this desire is the impetus of our sharing our work to date with fellow researchers now.

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