Article Title: Cartographies of Cognition: Charting the Links Between Mathematics Content and Pedagogy Concepts

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Keywords:
General School Subject:
Specific School Subject:
Students:

Macintosh File Name: vonMinden-Mathematics
Release Date: 10-1-97 A, 11-5-97 B

Editor: Abrams, Robert
Publisher: The Meaningful Learning Research Group
Publisher Location: Santa Cruz, CA
Volume Name: The Proceedings of the Fourth International Misconceptions Seminar - From Misconceptions to Constructed Understanding
Publication Year: 1997
Conference Date: June 13-15, 1997

Contact Information (correct as of 12-23-2010):

Web: www.mlrg.org

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Cartographies of Cognition: Charting the Links

Between Mathematics Content and Pedagogy Concepts

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The authors express appreciation to Barbara Beyerbach, Rayne Dennison, Lynn Housner, John Paterson, Richard Strasburger, and Judy Werner for their contributions to the investigation. Thanks are given to LuAnn Sturms and Ann Crabtree for manuscript preparation. This article is from the dissertation of the first author. For additional information, contact Department of Educational Psychology and Foundations, West Virginia University, Morgantown, WV 26506-6122.
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Abstract

The foci of the current study were to (a) explicate the nature of the links between content knowledge in the math domain and the pedagogical reasoning and actions proposed in teaching suggestions for problems containing a core concept set and (b) measure the adequacy of concept maps versus Pathfinder networks as structural knowledge elicitation techniques for the content concepts in a domain. Knowledge structure representations of university mathematicians, math educators, high school teachers, middle school teachers, and elementary school teachers were compared. Because of their training in both mathematical content and pedagogical concepts, math educators were expected to integrate these domains into a coherent pedagogical content knowledge structure. This was the case. Although mathematicians possessed integrated content knowledge structures, they tended to represent teaching as transmission of knowledge and learning as accumulation of knowledge. Math educators, high school teachers, elementary teachers, and in most instances, middle school teachers, appeared to conceptualize teaching as facilitation of conceptual change and learning as an interactive process. In addition, Pathfinder network analysis showed concept maps to yield more logically coherent representations of content knowledge structure than did similarity judgments of all pairwise comparisons.

Cartographies of Cognition: Charting the Links
Between Mathematics Content and Pedagogy Concepts

To better understand the teaching of mathematics, it is important to know how teachers of mathematics organize the conceptual relationships of content, the relationships of pedagogical concepts, and perhaps, most importantly, how teachers might connect these two sets of concepts (Ball & McDiarmid, 1990; Carpenter, 1989; Glaser, 1984; Hiebert & Lefevre, 1986; Hiebert & Carpenter, 1992). Carpenter (1989) suggested that the critical question for research on teachers' thought processes, knowledge structures, and problem solving asks ". . . what the nature of teachers' knowledge is and how it influences their instructional decisions" (Carpenter, 1989, p. 190).

Shulman (1986; 1987) commented that content knowledge and pedagogical strategies necessarily interact in the minds of teachers. He proposed that the teaching knowledge base includes content knowledge and pedagogical knowledge integrated in pedagogical content knowledge. This synthesis is teachers' special form of professional understanding, the distinctive architecture for teaching. It is the "blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners and presented for instruction" (Shulman, 1987, p. 8). Shulman and Quinlan (1996) reiterated this theme and reviewed evidence for differences across subject-matter areas in the nature of pedagogically powerful representations. For example, mathematics and social studies are quite different subjects. A skilled teacher who has deep understanding of both of these disciplines uses subject-specific strategies for designing and improvising instruction when teaching one or the other. These approaches to building on prior knowledge of the learners and toward authentic teaching and knowing go way beyond a generic set of pedagogical skills. Depth of knowledge of the content of mathematics or social studies is a necessary prerequisite to good teaching, and general teaching skills are desirable, but without pedagogical content knowledge specific to the domain, critical features of the necessary pedagogies of the subject matter are missed.

The same theme was addressed by Fennema and Franke (1992) who proposed that the measurement of teachers' knowledge of mathematics must relate the organization of their knowledge of mathematics to the organization of their teaching of mathematics. Schroeder and Lester (1989) determined that as individuals increase connections among mathematical ideas, they are (a) able to relate a given mathematical idea to a greater number or variety of contexts, (b) relate a given problem to a greater number of the mathematical ideas implicit in it, or (c) construct relationships among the various mathematical ideas embedded in a problem. What characterizes connections among (a) concepts of mathematics, (b) pedagogical knowledge, and (c) pedagogical content knowledge for mathematics teachers? In order to select a measure to assess these connections, one might consider the methods that have been employed to elicit knowledge structures. Early attempts to do so used semantic proximity techniques such as word association and similarity judgments tasks (e.g., Fenker, 1975; Shavelson, 1972; Wainer & Kaye, 1974). Similarity judgments, in which all possible pairs of concepts are presented, and participants are asked to assess the degree of similarity (highly related to unrelated) between each presented pair, have remained popular (e.g., Acton, Johnson, & Goldsmith, 1994; Gonzalvo, Canas, & Bajo, 1994). A triangular matrix of interconcept distances is then submitted to cluster analysis or multidimensional scaling. An alternative method is that of pathfinder analysis (Schavaneveldt, 1990) which uses a form of multidimensional scaling to yield spatial representations of cognitive structures. Such techniques have been demonstrated to be effective in differentiating low and high achieving students (e.g., Wainer & Kaye, 1974).

Another approach to eliciting structural knowledge is the use of concept maps (Novak & Gowin, 1984). Concept maps have also been effective in assessing learning gains and maturation of synthetic, integrative thinking (e.g., Beyerbach, 1988; Jones & Vesilind, 1996; Morine-Dershimer, 1993). The maps allow for the depiction of both hierarchical (inclusive) and web-like horizontal (comparative) links (e.g., Beyerbach, Smith, & Thomas, 1992; Markham, Mintzes, & Jones, 1994; Novak & Gowin, 1984). The constructive, generative mapping process may have the advantage of allowing mappers to be reflective while dealing with relationships among concepts presented simultaneously as opposed to an elicitation procedure, such as a similarity judgments task, assumed to be "minimally dependent on direct conscious access of knowledge" (Goldsmith, Johnson, & Acton, 1991, p. 94). During the latter, subjects must judge semantic distance between isolated concept pairs. In both concept mapping and similarity judgments tasks, however, semantic distance between concept nodes can be determined. Arguing for the centrality of proximity-based reasoning, Hirst (1991) reiterated that ". . it always comes back in some way to semantic distance as measured by some kind of physical distance in a network" (p. 23). In spreading activation theory of semantic knowledge, the strength of the connection between two concepts in a network (link strength) has been a cornerstone (Anderson, 1983).

The intent of the present investigation was to measure semantic distances among mathematics concepts and to examine knowledge structure representations of university mathematics professors (mathematicians), university math methods professors, and public school mathematics teachers (elementary, middle, and high school). The targeted knowledge structure representations were mathematics teachers' subject matter knowledge, their pedagogical decisions and suggestions, and possible links between them. An inseparable concern of the present work was the continuing research question of how such knowledge structures may be validly assessed. Accordingly, comparison was made for knowledge structures about mathematics content by (a) concept maps (in which all concept names are simultaneously available) and (b) similarity judgments (in which each pair of concept names is considered in isolation). Further elaboration of links between the core concept set and pedagogical reasoning was elicited by means of (a) concept identification of the concepts embedded within a set of problems and (b) their thoughts about teaching those problems.

With respect to Shulman's proposed model of pedagogical reasoning and action, the tasks included in the present research illuminate the links to comprehension (purposes, subject-matter structures, ideas within and outside the discipline) and transformation (preparation, representation, selection, and adaptation and tailoring to student characteristics) components of this model. Comprehension of the content concepts and purposes of mathematics education are necessary but not sufficient to differentiate teachers from non-teachers. If, as Shulman (1987) argued, "the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy . . ." (p. 15), then responses of math educators, because of their mathematics and pedagogical training and experience, should provide the clearest window for observing this nexus. Professors of math education engage in advanced educational scholarship and have continuing involvement with the literature and research outcomes in mathematics and pedagogy. They have knowledge of the content, knowledge of pedagogy, and rich interconnections. Thus, university math methods professors may show the most inclusive and widely-applicable wisdom of practice.

In addition to math educators, deep knowledge of math content should also be evident for university mathematics professors and high school teachers because of their levels of training and application (Nitko, 1989). While middle school teachers and elementary teachers may have somewhat more limited knowledge of content concepts, they, as well as the high school teachers, would likely know their own territory of teaching. Their wisdom of practice should allow them to smoothly navigate the landscape of teaching at their levels. Able, experienced teachers develop instructional strategies that link the content to the pedagogy for effective teaching. Elementary school, middle school, and high school teachers should be more apt at forging these connecting links for their grade levels than should university mathematics professors. The multiple tasks used herein were designed to assess these teachers' comprehension of subject-matter structures and transformation for the purposes of instruction. Further, the contrast of the use of a paired-comparisons test (semantic similarity judgments about pairs of content concepts presented in isolation) versus a concept mapping task (semantic similarity judgments about a set of simultaneously available concepts) will allow speculation about the adequacy of knowledge elicitation techniques and their subsequent structural representations.

Method

Participants and Design

The fifteen participants were three university-level mathematicians, three university-level mathematics methods educators, three high school math teachers (grades 9-12), three middle school teachers (grades 5-8), and three elementary school teachers (grades 1-4). The university mathematicians (32, 23, and 17 years experience) all (a) taught graduate and undergraduate students, (b) had no public school teaching experience, and (c) were unfamiliar with NCTM Standards. The university math educators (44, 15, and 12 years experience) all (a) had undergraduate teacher preparation responsibilities, (b) had public school teaching experience, and (c) were familiar with NCTM Standards. The high school math teachers (25, 16, and 15 years experience) all (a) taught a range of grades and math courses and (b) were familiar with NCTM Standards. Two had a bachelor's degree, and one had a master's degree. The middle school teachers (28, 23, and 14 years experience) currently taught grade 8, grade 5, and grade 8, respectively and had some familiarity with NCTM Standards. All three had a master's degree. The elementary school teachers (22, 17, and 16 years experience) currently taught grade 2, grade 3, and grade 2, respectively. Two of the teachers were familiar with NCTM Standards, and one was not. Two had a bachelor's degree, and one had a master's degree. The public school teachers all had responsibility for teaching math. Participation was voluntary. Each participant was offered a $30.00 stipend.

The focus of the study was to learn about the multiple perspectives held by teachers at different levels of practice on a core set of concepts. The National Council of Teachers of Mathematics (1989) Standards (NCTM Standards) included the 12 content concepts. These concepts are central threads across the curriculum, appear in earlier work about content structure in mathematics (e.g., Shavelson, 1974), and are stressed in the NCTM Standards. Thus, the pedagogical content knowledge (Shulman, 1987) describing the critical links between content and pedagogy were sought through a series of tasks. These tasks required mathematics teachers to make explicit their understandings of content concept links and the knowledge web that includes the teaching of those concepts at their own level of practice. Current scholarship about mathematics education, with a focus on teaching, has suggested that as individuals increase connections among mathematical ideas, they are (a) able to relate a given mathematical idea to a greater number or variety of contexts, (b) relate a given problem to a greater number of mathematical ideas implicit in it, or (c) construct relationships among the various mathematical ideas embedded in a problem (Lester, Masingila, Mau, Lambdin, Pereira dos Santos, & Raymond, 1994; Schroeder & Lester, 1989).

Procedure

Following an initial interview, each teacher was presented with a series of four tasks. These tasks were designed with a view toward providing an optimal context for observation of their knowledge structures and pedagogical decisions.

Task 1 required each participant to generate a concept map of the core concept set. A concept map is a schematic representation in two dimensions of the links between concepts in a particular knowledge domain. Participants arranged 12 content concepts that were provided, printed on oval cardboard discs (3-5 cm long), on a piece of white paper (56 cm by 56 cm). The 12 content concepts were binary operation, element, finite/infinite, ordered pair, set, associativity, commutativity, inverse, identity, function, distributive law, and estimation. Participants were instructed, in part, as follows.

> For this task, I would like you to think about mathematics. The twelve discs you see before you are labeled with mathematics content concepts. Take as much time as you want to look at them and their definitions on these papers. When you are ready, I would like you to arrange them (slide them around) on this big piece of paper to show the relationships of these concepts. Put close together the ones you see as highly related. Put far apart the ones that you think are not related. I'm going to tape the labeled discs to the paper. After that, we're going to draw lines to show which ones are connected. I will also ask you to describe for me your reason for connecting two concepts with a particular line. Later, I will measure the lengths of those lines to determine the distances among your concepts. Now, let me demonstrate what I mean by using the discs labeled with some mammal concepts.

A one paragraph definition including examples was provided on paper for each content concept. The participant could refer to these definitions and examples as desired throughout this concept map task. In drawing links with ruler and pencil, participants were asked to provide rationale for linking the concepts.

In Task 2, participants compared the provided content concepts on their map (and definition sheets) to nine word problems that were presented. The selection of the nine mathematics problems was based (a) on their appropriateness for instruction of the set of twelve mathematics content concepts and (b) on the NCTM Standards. The participants considered the strength of relationship between each problem and the content concept and assigned a value of 1 (problem and content concept not related) to 9 (problem and content concept highly related). They were instructed, in part, as follows, and these instructions were printed on the papers with the word problems.

> Below is a list of problems. Read each problem, one at a time, and decide whether or not it is related to any of the content concepts on your map. Once you have decided whether or not a relationship exists to any concept(s), I'll write the number you tell me next to that concept on your map. That is, decide how strongly a concept is related to the problem. The scale is 1 = not related to 9 = highly related.

The problems involved either one, two, or three of the content concepts. For example, the problem that concerned sharing pizza involved the content concepts of inverse and binary operation. The nine problems and the concepts embedded therein are reported in Table 1.

Insert Table 1 about here.

In Task 3, participants indicated their proposals for teaching each of the problems to a learner at any given level of their choice. For example, a mathematician may have proposed a method applicable to college students for teaching a problem involving the content concepts of inverse and binary operation. They were instructed, in part, as follows.

> That was fine. Now, I would like you to propose how you would teach these problems to the learners you work with. You can adapt the type of problem and the concepts involved in it to any level of student (high school, college, grade school). Start with Problem A. Tell me what alternatives (e.g., procedures, models, strategies, rules, representations, etc.) for teaching this type of problem you might consider and how you might decide what to choose. How would you do it?

In Task 4, participants judged the degree of relatedness between all pairwise comparisons (n=66) of the 12 mathematics content concepts. Each pair was rated for relatedness using a 1-9 scale, with 1 representing little or no relationship and 9 representing a strong relationship. Pairs were presented and responses recorded via computer. The participant viewed the instructions on the computer screen as the experimenter read them. All possible pairs of concepts were presented in an independent random order for each participant.

Results

<u>Task 1</u>

Do teachers at different levels organize concepts of mathematics content in different ways? When provided with the 12 content concepts, the teachers arranged cardboard discs with the concept names written on them to form concept maps. One such map is presented in Figure 1. The distance between any pair of concepts was measured in centimeters, if a link between two concepts had been drawn by the teacher. For example, in Figure 1, the concepts of associativity and function are indirectly linked through inverse and ordered pair or through ordered pair. The measurement was taken directly from the associativity disk on the map to the function disc. Even though a curved link might have been drawn to avoid other content concept discs, measures were taken in a straight line. Scaled distance scores (range 1 to 9) were calculated based on the longest raw distance for a connected (linked) pair of concepts (e.g., distributive law to set in Figure 1). The scale for each teacher was determined by dividing the longest raw distance by 8 and creating exact intervals. Concept pairs that were not linked were assigned a scaled score of 1 (e.g., estimate in Figure 1).

Insert Figure 1 about here.

To determine the extent of relationships among the five groups of math teachers, Pearson correlation coefficients were computed. A mean for the three participants in each of the five groups on the scaled distance scores was first determined for each of the 66 comparisons (Concept 1 to Concept 2, Concept 1 to Concept 3 . . . Concept 11 to Concept 12). Thus, there were 66 interconcept distance mean scores for the mathematicians, 66 interconcept distance mean scores for the math educators, and so on for the five groups. The correlation between mathematicians and math educators, for example, involved 66 scaled distance score means for each group. As displayed in Table 2, mathematicians showed closest agreement with math educators and least agreement with elementary school teachers. Math educators showed closest agreement with mathematicians and least agreement with elementary school teachers. High school teachers showed closest agreement with math educators and least agreement with elementary school teachers. Middle school teachers showed agreement with mathematicians, math educators, high school teachers, and elementary school teachers (these four correlations were not significantly different from each other). Elementary school teachers showed closest agreement with high school teachers and least agreement with mathematicians.

Insert Table 2 about here.

Discriminant analysis indicated that all 15 participants could be correctly placed in their groups based on 7 of the 66 distances. These most discriminating links, in order of their contributions (most to least) to the discrimination, were binary operation to infinite/finite, inverse to function, function to ordered pair, function to estimate, identity to ordered pair, commutativity to function, and commutativity to ordered pair. Thus, this combination of seven distances from the concept maps was sufficient to accurately identify each of the 15 teachers as a mathematician, math educator, high school teacher, middle school teacher, or elementary school teacher.

Task 2

Are there differences among teachers' ratings of the strength of relationships between math word problems and content concepts? As described, each of the nine word problems had an embedded content concept or concepts. Of the 18 occurrences of the content concepts in the problems, math educators identified an average of 11/18 (61%), high school teachers 10.7/18 (59%), mathematicians 9/18 (50%), middle school teachers 8.3/18 (46%), and elementary teachers 7/18 (39%). Correct identification was defined as a rating of 8 or 9 (on a nine-point scale) for a particular content concept embedded in that particular problem. The most accurately identified content concepts (identified by either all 15 or by 14 of the 15 participants) were commutativity, associativity, distributive law, and estimate. The least accurately identified content concepts were identity and inverse. Content concepts yielding the most discrepancies between groups were function (which occurred in 4 of the 7 discriminating links in Task 1), ordered pair (3 of the 7 discriminating links in Task 1), element, and binary operation (which occurred in the link with greatest contribution to the discrimination among groups of teachers in Task 1). Thus, the content concepts showing discrepancies among groups of teachers as recognized in the word problems (Task 2) also were strongly represented in the discrimination based on concept maps (Task 1).

All of the groups were accurate (67% or more) at identifying associativity, commutativity, distributive law, and estimate. All of the groups had difficulty (33% or less) at identifying the concepts of inverse and identity. All groups except elementary teachers demonstrated competence (67% or more) at identifying function and finite/infinite. Mathematicians and math educators were apt (67% or more) at identifying ordered pair and element. Elementary teachers also were apt at seeing element embedded in the problems. For both binary operation and set, the math educators and high school teachers were most proficient at identification.

Task 3

Is there congruence among participants about their pedagogical reasoning related to the concepts within word problems? From the transcribed tape recordings, each teaching suggestion was tabulated. These teaching recommendations are too numerous to be included herein but may be obtained from the first author. The recommendations were sorted into categories in accord with procedures described by Knafl and Howard (1984) and Knafl and Webster (1988). The most common suggestions were to (a) use concrete objects (including manipulatives) or representations of objects, (b) use illustrations (examples and non-examples), (c) use direct didactic methods such as stepwise telling, (d) use a constructivist approach such as guided discovery and activities allowing for individual differences, (e) read problem carefully or restate the problem, and (f) teach explicitly the concept embedded in the problem.

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Insert Table 3 about here.

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The predominant teaching strategy (Task 3) related to the predominant concept identified in each problem (Task 2) are summarized in Table 3. In Problem 1, commutativity was the predominant identified concept for all groups, and use of objects or representations was the predominant suggested strategy for all groups. In Problem 2, set was the predominant identified concept for all groups, and use of objects or representations was the predominant suggested strategy for all except mathematicians. In Problem 3, ordered pair was the predominant identified concept for mathematicians and math educators, and constructivist or individualized activities was the predominant suggested strategy for all except mathematicians. In Problem 4, binary operation was the predominant identified concept for only middle school, and use of objects or representations was the predominant suggested strategy for all except mathematicians. In Problem 5, associativity was the predominant identified concept for all groups, and constructivist or individualized activities was the predominant suggested strategy for math educators, high school, and elementary school. In Problem 6, infinite/finite was the predominant identified concept for all groups, and illustrations was the predominant suggested strategy for all groups. In Problem 7, function was the predominant identified concept for all except elementary school, and use of objects or representations was the predominant suggested strategy for all except mathematicians. In Problem 8, distributive law was the predominant identified concept for all groups, and use of objects or representations was the predominant suggested strategy for math educators, high school, and elementary school. In Problem 9, estimate was the predominant identified concept for all groups, and telling/stepwise was the predominant suggested strategy only for high school. Thus, from these findings and the summary in Table 3, there was most commonality in teaching suggestions among the public school teachers and math educators and less so with mathematicians.

Task 4

Does the use of a similarity judgments task elicited through all pairwise comparisons (content concept pairs presented in isolation) result in knowledge-structure representations that differ from those elicited through a concept mapping task (content concepts presented simultaneously)? In this final task, participants assigned a similarity judgment score to each pair of content concepts (e.g., binary operation, commutativity) presented on a computer screen. The scores ranged from 1 to 9 indicating unrelated to highly related concepts. For analysis, these similarity judgments were arrayed in a triangular matrix for each participant (Task 4). A second triangular matrix was constructed for each participant to indicate the scaled score distances derived from the individual's concept map (Task 1). These scores also ranged from 1 to 9. These two triangular matrices for each participant, one from all pairwise comparisons and one from the concept map, were each submitted to Pathfinder analysis. Pathfinder analysis extracts underlying latent structure from such a triangular matrix to generate a network representation of the derived knowledge structure.

Insert Figure 2 and Table 4 about here.

Pathfinder analysis computes coherence. Coherence may be thought of as triangulation in logic. If A is highly related to B, and B is highly related to C, then A should be related to C. As shown in Table 4, coherences are consistently higher for concept maps than for similarity judgments of all paired comparisons. Participants apparently are better able to give coherent cognitive structure representations when all content concepts are present in two-dimensional space (concept map) than when pairs of content concepts are compared in isolation from the others (pairwise comparisons). In addition, Pathfinder representations generally showed more interrelation of concepts for concept maps than for pairwise comparison (e.g., Figure 2).

Discussion

When provided with a given set of 12 mathematics content concepts (e.g., commutativity, function, distributive law), the prediction was that those having most extensive mathematical training and experience (mathematicians, math educators, and high school teachers) would produce concept maps that were most alike. Highest correlations occurred between the maps of the mathematicians and the math educators and between the maps of the high school teachers and the math educators. Based on Schroeder and Lester's (1989) findings, the expectation for math educators' ability to relate given mathematical ideas to a variety of problem contexts and a given problem to a greater number of implicit ideas, was fulfilled. A counter-intuitive finding, however, was the superior performance of high school teachers as compared to mathematicians in identifying math content concepts in a variety of contexts. Even though middle and elementary school teachers lagged behind in their ability to recognize embedded concepts, the public school teachers and the math educators exhibited consistency in their pedagogical reasoning and proposed strategies for instruction.

The math educators as well as their experienced public school colleagues offered congruent pedagogical suggestions for teaching Problems 1, 2, 3, 4, 6, and 7. On Problems 5 and 8, math educators, high school teachers, and elementary teachers proposed similar teaching strategies. Thus, math educators, high school teachers, and elementary school teachers were in league on eight of the nine provided problems, and all four groups (including middle school teachers) were consonant in their recommendations on six of the nine problems. Mathematicians joined this pattern only on Problems 1 and 6.

Mathematicians' concept map representations of content knowledge were most highly correlated with those of math educators (Task 1). Mathematicians were behind math educators and high school teachers in accuracy of identifying content concepts in word problems (Task 2). And, as noted, they differed widely from the math educators and public school teachers in teaching recommendations (Task 3). An example of math educators' typical strategies for teaching follows. "Use technology as a vehicle for guided discovery of inductive reasoning-type approaches. Students generate data, analyze data, and plot data . . . Perturb by exemplification, counter examples, open-ended questions centering on, or coming out of, applications, to motivate their desire or even need to know more." Such responses exemplify coherent pedagogical content knowledge (e.g., Ball, 1991; Fennema & Franke, 1992; Hiebert & Lefevre, 1986; Lampert, 1989; Shulman, 1986; 1987; Shulman & Quinlan, 1996).

When questioned about how they would propose teaching each of these problems, the elementary school group offered the greatest number of teaching suggestions and confined their responses to applicability for the early grades. Math educators offered suggestions that encompassed the widest range of learners. The teaching suggestions of the math educators and the public school teachers implied a conception of learning as active process and teaching as facilitation of conceptual change (Prosser, Trigwell, & Taylor, 1994). In response to one of the word problems, an elementary school teacher said:

> Take a survey. Chart [their answers] . . . Ask lots of questions; then get them to ask questions . . . It makes the thinking process a little bit different . . . I do ask them to verbalize about math . . . They have to think about it in a different way . . . They have to explain it to someone else because -- and you know we have some children that -- if you just look at their paper seem to be very good at math, but they can't tell you why . . . But I do want them to be able to share and say okay how did you do that. Sometimes they can do that. Sometimes not. That's hard to do.

High school teachers had deep understanding of content knowledge and could identify concepts in context. They assumed a student-oriented stance to teaching (similar to that of the math educators). One high school teacher justified rearranging the sequence of instruction for two of the word problems this way.

> In 1, it's a number property that you can --they can-- experiment with, and then, when they notice that sometimes it works and sometimes it doesn't work -- we're going to give it names when it works. But 2 -- in order for us to classify these things, we already have to have, because it's a description -- it isn't just a property that you can mess around with . . . you can define it first and then mess with it because . . . you're going to put different definitions together

and so, OK, I have some of this and some of this so this is a rectangle . . .

Mathematicians were effective at organizing important mathematical concepts, but they were not particularly accurate at identifying the concepts in word problems. Their pedagogical knowledge was less learner focused and more algorithmic. They appeared to characterize learning and teaching as more didactic and unidirectional, the transmission of information to more passive learners (Prosser, Trigwell, & Taylor, 1994). For example, typical responses of mathematicians follow.

> Tell them matrix multiplication is not commutative. Give them a proof (Response 1). Explain the way I understand it . . . I kind of like the way I try to explain things even though it's not very effective . . . to me it's very important how I see things, how I understand things or conceptualize things and communicate that to them . . . (Response 2).

It appears that the crucial juncture for developing sophisticated pedagogical content knowledge lies in the transformation component of Shulman's (1987) model, as exemplified in the present study by math educators. Extending the present research, classroom-based video recording of teaching by mathematics educators could be parsed into pedagogical cases. Building a database of cases would afford educators the opportunity to study and conduct fine-grained analyses of these teachers' actions. In addition, a hypermedia database would provide teacher educators with easily accessible instances of teaching that are conducted in accord with that stressed by scholars of mathematics and educational reform.

Finally, a question arose from the comparison of the concept mapping elicitation technique with that of the similarity judgments technique. If one of the intents of the reform of schooling in the United States is promotion of reflective practitioners and students, then a method that assumes "minimal direct conscious access of knowledge" (Goldsmith, Johnson, & Acton, 1991, p. 24) appears contradictory. The value of concept mapping appears to lie in (a) its assumption of generative and synthetic processes to elicit and represent implicit knowledge about the interrelationships among concepts in a domain and (b) its simultaneous presentation of an array of concepts that may be manipulated. If the ability to perceive the connections between concepts is a necessary component for constructing sound pedagogical content knowledge, then techniques that promote more logically coherent representations of those connections appear promising.

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Table 1

Provided Word Problems and the Embedded Content Concepts[a]

1. A bowl holds 7 large apples and 3 small apples. How many apples are there altogether? Write the number sentence in 2 ways to solve this problem. (Commutativity, Binary Operation, Identity)

2. Your category box holds different objects. Take all of the large, round, green objects and place them in a pile. (Element, Set)

3. Birthday hats pictograph with Number of Birthdays on the Y axis and Months on the X axis. Problem Statement: Which month has the most birthdays? How many birthdays are in that month? (Ordered Pair)

4. There were one and a half pizzas left after the party. Bill's parents told him that he must share the pizza equally with his brother and sister. How much pizza will Bill and his brother together receive? (Inverse, Binary Operation)

5. Mary had an addition problem: (5 + 7) + 3. Can you rewrite the problem to make this simpler to add? (Associativity, Binary Operation)

6. You have all shared the names of the largest numbers you can imagine. Later, one of the students named an even larger number. What do you think this means? (Infinite/Finite, Set)

7. Two Function boxes, the first depicting input of 3 and output of 8, and the second depicting input of 7 and output of 12. Problem Statement: What happened to the 3 and the 7 inside the box? (Function, Binary Operation)

8. Two rectangles, the first depicting 5 cells X 4 cells and labeled 4 X (3+2) = *, and

Table 1 (continued)

the second depicting 3 cells X 4 cells and 2 cells X 4 cells and labeled (4X3) + (4X2) = ! Problem Statement: Study picture A, then find *; Study picture B, then find !. Compare * and !. What do you discover? (Distributive Law, Binary Operation)

9. Look at the multiplication problem below. Without using a calculator or a pencil and paper, write an approximate answer to the problem beside it.

300 X 300 = 90,000 297 X 305 =

(Estimate, Binary Operation)

[a] The embedded concepts were not printed on the problem page presented to the participants.

Table 2

Correlations [a] of the 66 Scaled Distances (Means Across the Three Participants in Each Group) from Concept Maps

	Mathe- maticians	Math Educators	High School	Middle School	Elementary School
Mathe- maticians	X				
Math Educators	0.735	X			
High School	0.528	0.629	X		
Middle School	0.643	0.625	0.597	X	
Elementary School	0.241	0.380	0.555	0.466	X

[a] All Correlations $\underline{p} <.01$ except $\underline{r} = 0.241$, $\underline{p} <.05$.

Table 3

Predominant Teaching Strategies (Task 3) Related to Predominant Mathematics Concepts Identified in Problems (Task 2)

Problem	Mathematicians	Math Education	High School	Middle School	Elem. School
Problem 1					
Concept	commutativity	commutativity	commutativity	commutativity	commutativity
Strategy	objects/reps.	objects/reps.	objects/reps.	objects/reps.	objects/reps.
Problem 2					
Concept	set	set	set	set	set
Strategy	set notation	objects/reps.	objects/reps.	objects/reps.	objects/reps.
Problem 3					
Concept	ordered pair	ordered pair	none	none	none
Strategy	restatement	construct./indiv.	construct./indiv.	construct./indiv.	construct./indiv.
Problem 4					
Concept	none	none	none	binary oper.	none
Strategy	read carefully	objects/reps.	objects/reps.	objects/reps.	objects/reps.
Problem 5					
Concept	associativity	associativity	associativity	associativity	associativity
Strategy	telling/stepwise	construct./indiv.	construct./indiv.	telling/stepwise	construct./indiv.
Problem 6					
Concept	infinite/finite	infinite/finite	infinite/finite	infinite/finite	infinite/finite
Strategy	illustrations	illustrations	illustrations	illustrations	illustrations
Problem 7					
Concept	function	function	function	function	none
Strategy	none	objects/reps.	objects/reps.	objects/reps.	objects/reps.
Problem 8					
Concept	distributive	distributive	distributive	distributive	distributive
Strategy	none	objects/reps.	objects/reps.	none	objects/reps.
Problem 9					
Concept	estimate	estimate	estimate	estimate	estimate
Strategy	none	none	telling/stepwise	none	none

None = no predominant (majority) identified Concept or Strategy.

Table 4

Coherences for Pathfinder Network Representations Elicited by Concept Mapping versus Similarity Judgments (All Pairwise Comparisons) for Mathematicians (M), Math Educators (ME), High School Teachers (HS), Middle School Teachers (MS), and Elementary School Teachers (ES)

Participant	Concept Map	Similarity Judgments
M1	.882	.416
M2	.911	.238
M3	.857	.255
ME1	.808	.570
ME2	.909	.112
ME3	.906	.430
HS1	.863	.367
HS2	.865	.516
HS3	.800	.572
MS1	.857	.284
MS2	.883	.839
MS3	.851	.553
ES1	.966	.708
ES2	.943	-.246
ES3	.857	.603

Figures

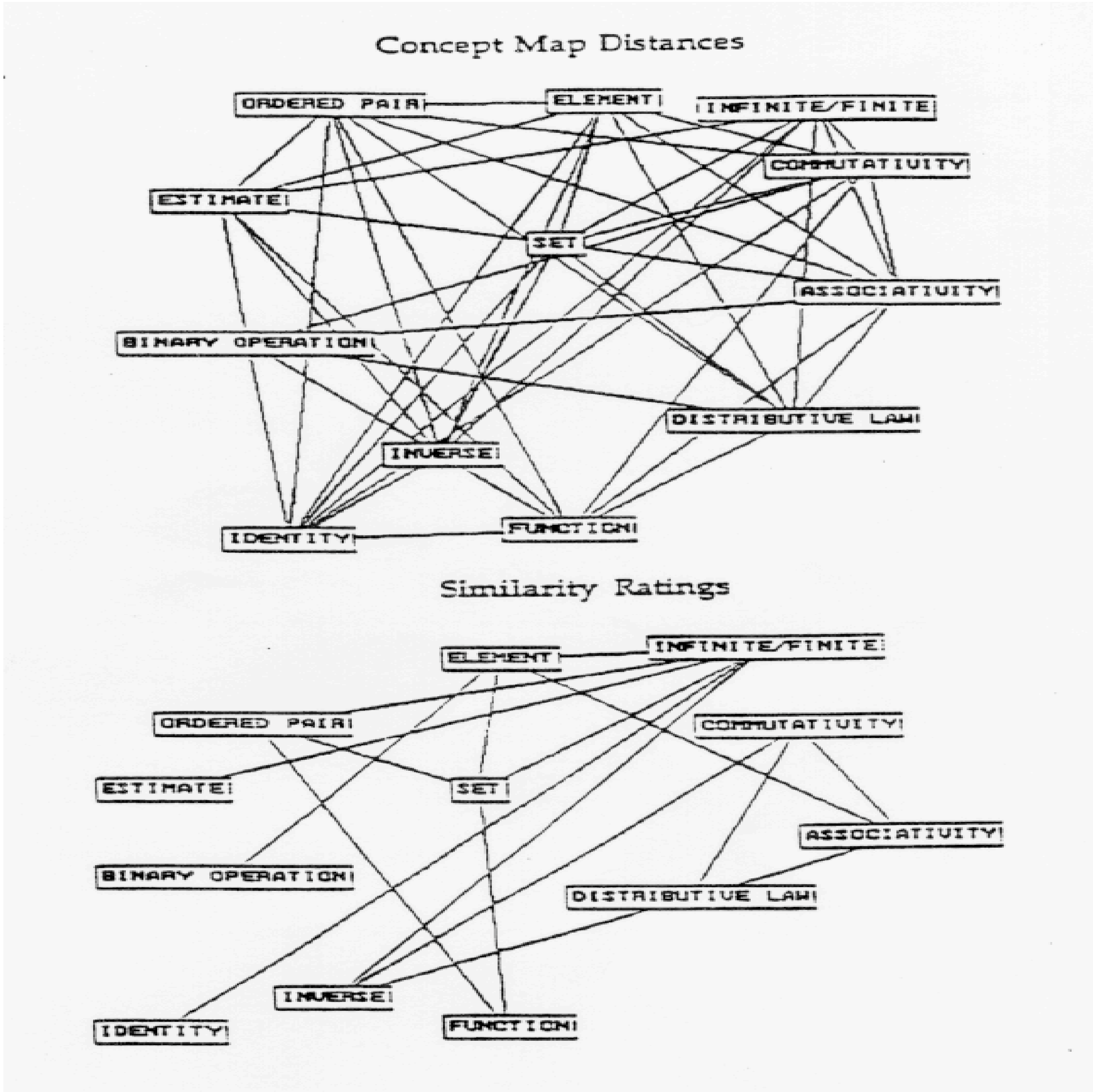

Figure 1. Sample teacher concept map.

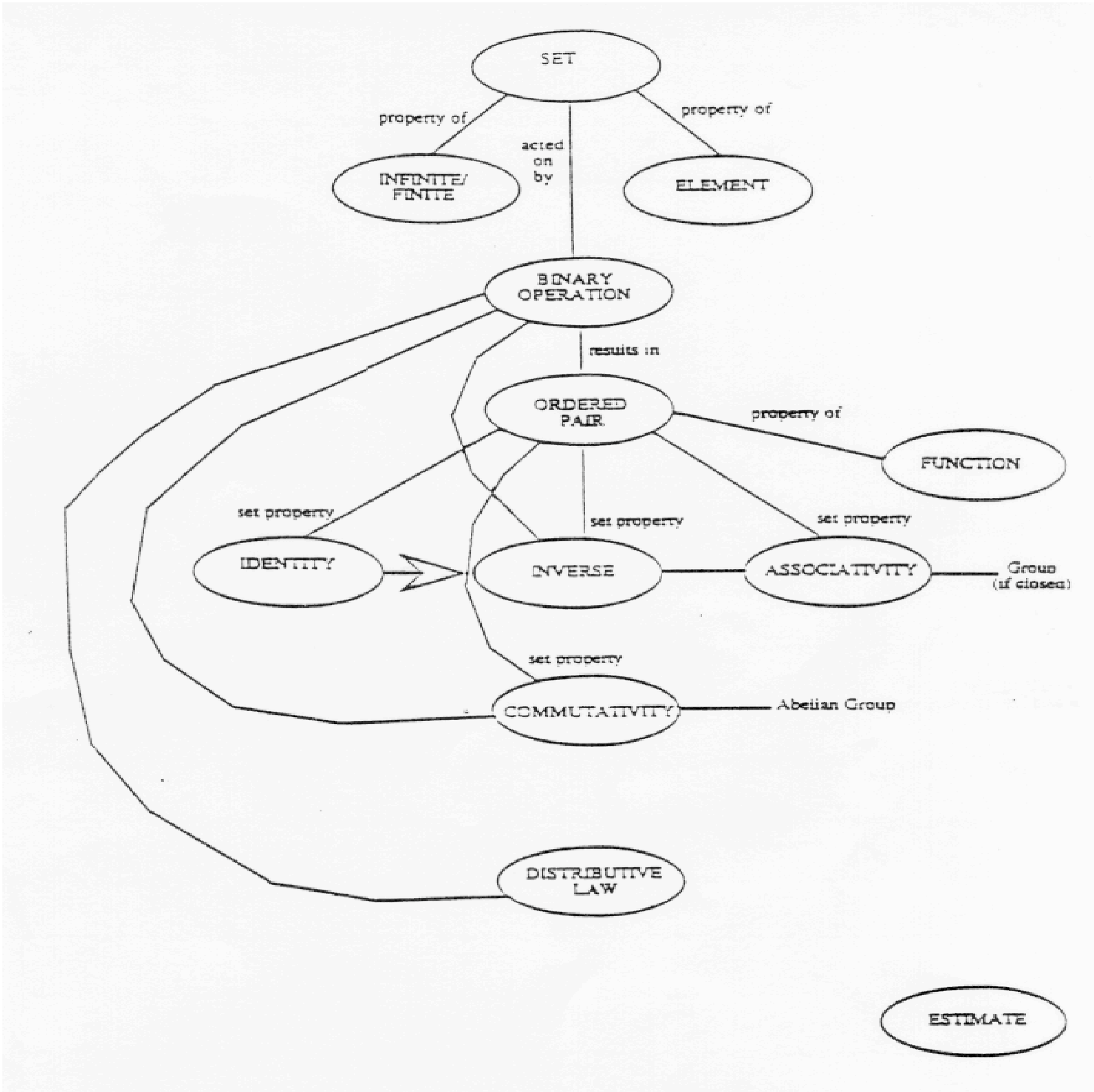

Figure 2. A math educator's Pathfinder network representations for the concept map (based on scaled distances) versus similarity ratings of all pairwise comparisons.