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Paper Title: **SOURCES OF MISCONCEPTIONS: THE CASE OF POWERS AND RADICALS**

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1- Interpreting radicals (when the index is greater than 2) either as powers or as square roots. This category had the highest frequency of misconceptions.

2-Applying rules of multiplication of powers.

3-Applying rules which are not related to the concepts of powers and radicals such as operations on negative numbers and simplified writing.

Results also showed that with the increased use of the rules across the different grade levels the frequency of their incorrect applications decreased.

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The Case of Powers and Radicals

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SOURCES OF MISCONCEPTIONS:
THE CASE OF POWERS AND RADICALS

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In the last two decades errors or misconceptions in mathematics have been the topics of many research studies (Radatz, 1979; Watson, 1980; Clements, 1980; Movshovitz Hadar, 1987; Blando et al 1989). While the focus of many studies was on children's errors in verbal problem solving (Hollander, 1979; Clement, 1980) researchers' interest shifted to classifying errors in arithmetic and recently to errors in advanced topics. Attempts had also been made to describe theoretical models for errors such as repair theory (Brown & Vanlehn, 1980) which considers that errors occur when students try to resolve their impasse by modifying a known procedure and incorrectly applying it to the task. The work of Resnick et al (1989) showed that errors derive from students attempts to integrate new material that they are taught with already established knowledge. The interference of prior knowledge with newly learned material was mainly investigated at the level of procedural knowledge. Because of the relationship between powers and radicals teachers encourage students to use their prior knowledge in powers to make computations in radicals. The present study aims at investigating students' knowledge of powers and its effect on their understanding of the concept of radicals and the operations on radicals. As for the method used, the present study is using both, a written test followed by diagnostic interviews to probe in depth the sources of students misconceptions.

METHOD

SUBJECTS

One hundred sixty eleventh grade students in the scientific track of the academic high school from six schools participated in the study. The schools were located in the capital Beirut, four of which were private schools and the other two were public schools. The concept of powers is introduced at the sixth grade level in the Lebanese program, while the concepts of square root and n^{th} root are introduced at the tenth and eleventh grade levels respectively.

INSTRUMENT

A written test on powers and radicals was designed to reveal misconceptions in (a) meaning of powers and radicals (b) relationship between powers and radicals and (c) operations on powers and radicals. The test consisted of tasks that required the student either to make simple calculations or to transform a mathematical expression from one form into another. The 36 different test items represented instances that would help in identifying sources of misconceptions. Table 1 shows a sample of the test items.

Table 1

Sample of the test items

Area of misconception / Item	Meaning	Relationship	Operations
1	Write in the form of power i) $2 \times 2 \times 2 \times 2$ ii) $a \times a \times a \times a \times a$	Write in the form of a radical i) $N = a^m \Leftrightarrow a = \ldots$ m is a natural number N is a real number ii) $36^{\frac{3}{4}}$	Write in the form of a single power i) $7^2 \times 7^0 \times 7^5$ ii) $3^2 \times 9^4 \times 27^5$
2	Calculate i) $\sqrt{25}$ ii) $\sqrt{(-4)^2}$ iii) $\sqrt[4]{16}$	Write in the form of a single power i) $\sqrt[4]{a}$ ii) $\sqrt[3]{a^2}$	Without making any computation write in the form of a single power i) $\frac{15^7}{8^7}$ ii) $\frac{3^6}{5^3}$ iii) $\frac{7^5}{7^4}$ iv) $\frac{(-8)^9}{(-8)^4}$ v) $\frac{12}{12^0}$ vi) $\frac{(-5)^{11}}{(-5)^{11}}$

PROCEDURE

The test was administered by the teachers of the different classes eight weeks after students completed the study of powers and radicals. The test was corrected by the author. An item was analyzed if 30% or more of the students gave it an incorrect solution. Both the final answer and the method of solution were considered in the classification of the misconceptions.

CLASSIFICATION OF MISCONCEPTIONS

Items were analyzed in a qualitative manner. Whether some or all the different steps for the solution were written by the student, the investigator's task was to identify the rule or rules that justified the students 'work. All incorrect rules used by the students as judged by the author were listed. An item was classified under a certain incorrect rule if 20% or more of the students used that incorrect rule to get their answer. Incorrect rules used by the students were recorded in each of the three groups of items i.e. items related to the meaning of powers and radicals, items related to the operations on powers and radicals and items related to the relationship between both concepts.

DIAGNOSTIC INTERVIEWS

Eighteen students were interviewed by the author, two weeks after the test was administered and corrected. One or two students using a classified incorrect rule were randomly chosen and interviewed individually and the interview was audio taped. In the interview the student was asked to reproduce the answer to a certain item so as to find out whether he uses the incorrect rule consistently. He was then asked to explain how did he arrive at the incorrect answer.

RESULTS

MEANING OF POWERS

As for the meaning of powers, results showed that 20% or less of the students gave incorrect answers to the items in this category except for the item 5^{-2} as shown in table 2.

Table 2

Percentage of incorrect answers and those using the same incorrect rule in the items related to meaning of power.

Item	Write in the form of power		Calculate						
	2x2x2x2	$axaxaxaxa$	100^3	425^0	-2^4	$(-1)^5$	219^1	10^{-3}	5^{-2}
Percentage of incorrect answers	2%	3%	20%	13%	4%	8%	1%	20%	35%

The answers to the item 5^{-2} were analyzed and it was found that 35% of the students either gave the answer 0.05 without writing any intermediate step or wrote the following:

$$5^{-2} = 5 \times 10^{-2} = 0.05$$

The author assumed that students have used the incorrect rule $a^n = a \times 10^{10}$.

MEANING OF RADICALS

As for the meaning of radicals results showed that 35% or more of the students gave incorrect answers to most of the items in this category as shown in table 3.

Table 3

Percentage of incorrect answers and those using the same incorrect rule in the items related to meaning of radicals.

Item	Calculate whenever possible					
	$\sqrt{25}$	$\sqrt{(-4)^2}$	$\sqrt[4]{16}$	$\sqrt[3]{-27}$	$\sqrt[4]{-20}$	$\sqrt[5]{-100000}$
Percentage of incorrect answers	5%	35%	46%	75%	50%	78%
Percentage using the same incorrect rule		20%	41%	70	70%	75%

The highest frequency of error was with items having a negative number under the radical. The response to these items was "impossible to find the root." which means they interpret it as a square root. The response to the item $\sqrt[4]{16}$ was 4 which means that it was also interpreted as a square root. The negative number under the square root was also misleading to many students where the response to $\sqrt{(-4)^2}$ was -4 i.e. the definition of a square root was not clearly understood by the students, their answers could be a result of defining $\sqrt{a^2} = a$.

RELATIONSHIP BETWEEN POWERS AND RADICALS

Table 4 shows that misconceptions were strongly marked when the transformation from power form to radical form involved fractional indices. The item $36^{\frac{3}{4}}$ was written as $\sqrt[3]{36^4}$ similarly $a^{\frac{-7}{2}}$ was written as $\sqrt[-7]{a^2}$. It seems that students confused the index of the radicand and the index of the radical sign. On the other hand students found it easier to transform a radical form into a power form.

Table 4

Percentage of incorrect answers and those using the same incorrect rule in the items related to the relationship between powers and radicals.

Item	Write in the form of radical			Write in the form of power	
	$A = a^m \Leftrightarrow a = \ldots$	$36^{\frac{3}{4}}$	$a^{\frac{-7}{2}}$	$\sqrt[4]{a}$	$\sqrt[3]{a^2}$
Percentage of incorrect answers	25%	60%	57%	16%	20%
Percentage using the same incorrect rule		29%	40%		

OPERATIONS ON POWERS

In this category, students task was not to make computations but to use multiplication and division rules of powers to write an expression in the form of a single power. Table 5 shows that students had more difficulty in applying the rules of division and multiplication when the item contained different bases and different indices.

Table 5

Percentage of incorrect answers and those using the same incorrect rule in the items related to operations on powers.

Item	Write in the form of a single power		Without making any computations, write in the form of a single power						
	$7^2 \times 7^0 \times 7^5$	$3^2 \times 9^4 \times 27^5$	$\frac{15^7}{8^7}$	$\frac{3^6}{5^3}$	$\frac{a^5}{b^5}$	$\frac{7^5}{7^4}$	$\frac{(-8)^8}{(-8)^3}$	$\frac{12}{12^0}$	$\frac{(-5)^{11}}{(-5)^{11}}$
Percentage of incorrect answers	16%	45%	35%	50%	30%	30%	40%	32%	34%
Percentage using the same incorrect rule		20%	38%	30%	35%	27%	29%	15%	25%

Students written work revealed that to write $3^2 \times 9^4 \times 27^5$ in the form of a single power they multiplied the bases and added the indices while in using the rules of division of powers students strategy was to transform the division into multiplication which they did correctly, but they applied again their incorrect rules of multiplication. An example is $\frac{3^6}{5^3} = 3^6 \times 5^{-3} = 15^3$. It is worth noting that rules of operations on negative numbers were applied regardless of the index of the number such as $\frac{(-8)^8}{(-8)^3} = 8^5$.

OPERATIONS ON RADICALS

Table 6 shows that out of the seven items only four were analyzed and of the four three revealed certain patterns of misconceptions.

Table 6

Percentage of incorrect answers and those using the same incorrect rule in the items related to operations on radicals.

Item	Write each in the simplest way possible					Without changing its value write each expression with index ...	
	$\sqrt[5]{4}\times\sqrt[5]{8}$	$\sqrt[5]{2}\times\sqrt[4]{2}$	$\frac{\sqrt[3]{2}}{\sqrt{2}}$	$\left(\sqrt[4]{b}\right)^3$	$\sqrt{a^2\times b}\times\sqrt{b}$	$\sqrt[12]{a^2}$	$\sqrt[3]{b^5}$
Percentage of incorrect answers	12%	60%	70%	50%	15%	25%	30%
Percentage using the same incorrect rule		30%	35%	40%			10%

Misconceptions in this category were the result of interpreting radicals as powers. Two patterns of incorrect answers were identified for the item $\sqrt[5]{2}\times\sqrt[4]{2}$. The first is multiplying the two radicals and adding their indices, and the second is keeping the radical and multiplying the indices. Both patterns were a result of interpreting radicals as powers. The same misconceptions apply to the division of radicals where $\frac{\sqrt[3]{2}}{\sqrt{2}}$ was written as $\sqrt[3-1]{2}=\sqrt{2}$. It also applies to powers of radicals where $\left(\sqrt[4]{b}\right)^3$ was written as $\sqrt[12]{b}$. Although 30% of the students were not able to write correctly $\sqrt[3]{b^5}$ with index 6, yet no pattern was revealed in students responses.

INTERVIEWS

As mentioned earlier individual interviews were done with eighteen students identified as using certain incorrect rules. The aim of the interview was to test whether the author's judgment of the students' use of incorrect rules was a right one, to find out why did a student use an incorrect rule, and whether he uses it consistently. Below are examples of interviews where A designates the author and S designates a student.

Example 1:

S_1 is a student identified as using the incorrect rule $a^n = a \times 10^n$.

A : Explain how do you get the value of 5^{-2}

S_1: I know that $5^{-2} = 5 \times 10^{-2}$

A : Please write it down.

S_1: I can do it on the calculator.

A : Is that what you did in the test ?

S_1: No, after I finished the test I checked the answer with the calculator $5 \times 10^{-2} = 0.05$

A : Why did you write 5×10^{-2}

S_1: Because we can write any number in the form of a power of 10.

A : Give me other examples.

S_1: $15^{-6} = 15 \times 10^{-6}$ --This rule applies always to negative powers.

Example 2:

S_2 is a student identified as using the incorrect rule $a^m . b^n . c^p = (abc)^{m+n+p}$.

A : How do you write $3^2 \times 9^4 \times 27^5$ in the form of a single power?

S_2: $3^2 \times 9^4 \times 27^5 = (3 \times 9 \times 27)^{11}$

A : Can you specify the rule you used?

S_2: Multiply the number (pointed to the bases) and add the powers(pointed to the indices).

Example 3:

S_3 is a student identified as using the incorrect rule $\frac{a^n}{b^n} = (a \times b)^{n-n}$.

A : How do you write $\frac{15^7}{8^7}$ in the form of a single power?

S_3: We can always move the power from the denominator to the numerator as such $\frac{15^7}{8^7} = (15 \times 8)^{7-7}$.

A : What rule did you use to write the last step?

S3: When the numbers(pointed to the bases) are different you

multiply them and add the powers(pointed to the indices).

A : How about $7^4 \times 7^3$

S3: Equal $(7 \times 7)^{4+3}$

A student who wrote $\frac{(-8)^8}{(-8)^3} = 8^5$ explained that to get rid of the minus sign he multiplies the numerator and denominator by a minus sign. When the author asked him whether he does this always he paused and said not when we have odd powers. As for the operations on powers and radicals the six interviewed students said that they can work with radicals the same way they do with powers.

It remains to mention that the meaning of radicals (when the index is more than 2) was not clear to five interviewed students when the radicand was negative, they interpreted it as a square root, while three students realized that $\sqrt[4]{16}$ is 2 and not 4.

INCORRECT RULES

Based on the written test results, and interviews a list of incorrect rules was developed as shown in table 7.

Table 7

List of incorrect rules

Incorrect rule:	Example:
$a^n = a \cdot 10^n$ (n is negative)	$5^{-2} = 0.05$
$\frac{(-a)^n}{(-a)^m} = a^{n-m}$	$\frac{(-8)^9}{(-8)^4} = 8^5$
$a^n \cdot a^m = (ab)^{n+m}$	$3^2 \times 9^4 \times 27^5 = 729^{11}$
$\frac{a^n}{b^n} = (ab)^{n-n}$	$\frac{15^7}{8^7} = (15 \times 8)^{7-7}$
$\sqrt{a^2} = a$	$\sqrt{(-4)^2} = -4$
$\sqrt[n]{-a}$ impossible to find (a is positive)	$\sqrt[3]{-27}$ impossible to find its root
$\mathrm{a}^{\frac{\mathrm{m}}{\mathrm{n}}} = \sqrt[m]{a^n}$	$\mathrm{a}^{\frac{-7}{2}} = \sqrt[-7]{\mathrm{a}^2}$
$\sqrt[n]{a} \times \sqrt[m]{a} = \sqrt[nm]{a \cdot a}$ or $\sqrt[\mathrm{n+m}]{a}$	$\sqrt[4]{2} \times \sqrt[5]{2} = \sqrt[20]{4}$ or $\sqrt[9]{2}$
$\left(\sqrt[n]{b}\right)^m = \sqrt[nm]{b}$	$\left(\sqrt[4]{\mathrm{b}}\right)^3 = \sqrt[12]{\mathrm{b}}$
$\frac{\sqrt[n]{a}}{\sqrt[m]{a}} = \sqrt[n-m]{a}$	$\frac{\sqrt[3]{2}}{\sqrt{2}} = \sqrt{2}$

CONCLUSION

Findings are clearly consistent with other recent research findings in the area of misconceptions. It confirms the hypothesis that students' prior knowledge interfers with the construction of new knowledge. In analyzing the list of incorrect rules used and the interviews done with the students three main sources of misconceptions emerge, a) misconceptions derived from students use of rules irrelevant to the concepts of powers and radicals such as the use of scientific writing and the use of operations on negative numbers . b) misconceptions derived from students use of the multiplication operation on powers. Although the concept of powers was clear to almost all students yet 45% were not able to apply correctly the rule of multiplication of powers when the bases were different and the indices were also different which shows that students do not give concecptual meaning to the rules c) misconceptions derived from students' interpretation of radicals with index more than 2 either as powers or as square roots. The highest percentage of misconceptions was in this category. In the Lebanese program square roots are introduced one year prior to the introduction of n^{th} root. As it is noticed up to 78% interpreted n^{th} roots as square roots especially when the radicand was negative. The percentage dropped to 46 when the radicand was positive. One would then assume that the failure to perform operations on radicals could be the result of unclear meaning of the concept of radicals which is not true in the case of powers. While students had no difficulty in transforming radicals into powers yet many of them did not use the strategy in performing operations on radicals, instead they multiplied and divided radicals the same way they did with powers using correct and incorrect rules of multiplication of powers. The above means that whether students know the meaning of powers and radicals or not, they apply incorrect rules when it comes to computation. The feasible assumption then is that students use rules without reference to its conceptual meaning which is consistent with Browns and VanLehn findings(1980). As for the consistency of using an incorrect rule, most interviewed students repeated the solutions given in the written test.

Results of the present study could be used by teachers to emphasize: 1) the multiplication rule of powers, 2) the difference between a square root and an n^{th} root and 3) the strategy of changing a radical into a power form before making computations.

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