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Author: Schmittau, Jean

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HISTORICAL AND CONCEPTUAL ANALYSIS AS FOUNDATIONS FOR CURRICULUM: MULTIPLICATIVE STRUCTURE AS A CASE

Jean Schmittau
State University of New York at Binghamton (USA)

William James wrote in "The World We Live In":

> Out of time we cut "days" and "nights", "summers" and "winters." We say _what_ each part of the sensible continuum is, and all these abstract _whats_ are concepts.
>
> The intellectual life of man consists almost wholly in his substitution of a conceptual order for the perceptual order in which his experience originally comes.

Perhaps no discipline exemplifies the conceptual order of which James writes better than mathematics, whose concepts cannot be detected at the surface of the world of form, but require the development of a theoretical mode of thought. Mathematics concepts are typically of the type designated by Vygotsky as "scientific" rather than "everyday"; they are most often the subject of school instruction rather than the result of environmental interaction; and unlike everyday concepts which can be spontaneously constructed, they require pedagogical mediation for their appropriation.

When pedagogical mediation is inadequate, learning is impeded. The nature and extent of pedagogical mediation is determined through a genetic analysis (Davydov, 1972/1990), which encompasses both an epistemological analysis of the sociocultural construction of the concept as an historical product, and a psychological analysis which determines the requisite actions through which the concept may be individually appropriated by the student. Such analyses, of course, have not characterized mathematics curriculum construction in the United States, which has been subject to the influences of a marriage of convenience between formalism in mathematics and behaviorism in psychology.

HISTORICAL AND CONCEPTUAL ANALYSIS IN THE CONSTRUCTION OF A CURRICULUM FOR THE CONCEPT OF REAL NUMBER

An excellent example is provided by the concept of number, one of the most fundamental in the mathematics curriculum. In the United States, our approach has been to build on children's spontaneously constructed counting sequences (since many children come to school with some ability to count, even if only by rote), with the result that after reinforcement through several years of exclusive emphasis, the counting numbers come to constitute the "real" numbers for the child. When fractions, which cannot be generated through the activity of counting, are introduced, the child's schema for number which by this time is firmly entrenched, must be reconstructed to encompass the new numbers. In practice, such a reconstruction constitutes a Herculean cognitive task which is rarely accomplished, prompting Skemp (1987) to comment that even among adults, there are few who really understand fractions.

Further, the necessity of introducing fractions into a schema for number predicated on cardinality results ultimately in an attempt to connect the two types of numbers formally, by defining the fraction as a quotient of two counting numbers. In my research with students from elementary school through the university, both in the United States and in the former Soviet Union, I have not found a single instance in which the psychological structure of a mathematics concept reflected a formal linkage with other concepts to which it was mathematically related (Schmittau, 1991a). In practice, it appears that either such concepts are meaningfully connected (that is, integratively reconciled within the cognitive structure) or they are the products of rote learning and hence, are not connected at all (See Ausubel, Novak, & Hanesian, 1978).

If even a rudimentary genetic analysis of the type advocated by Davydov were carried out, it would become obvious that building the concept of number on the results of counting is just about the worst way to teach the concept, and one that might be expected to continue to interfere with the meaningful learning of mathematics for many years to come (See Schmittau, 1988, 1991b). To understand this, however, it is necessary to consult the history of mathematics, which provides a cognitive record of the sociocultural construction of the real numbers.

The ancient Greeks, whose influence on the development of mathematics in the west has been pervasive, separated the products of counting from the products of measurement. They considered the former to be "numbers", but regarded the latter only as "magnitudes". Consequently, although they could quite precisely determine $\sqrt{2}$ as the diagonal of a unit square, and even the sum $\sqrt{2} + 1$ by the combination of the diagonal with the length of one of the sides, they did not consider these to be numbers. It required two thousand years to unite both the products of counting and the products of measurement into the concept of the real numbers, represnted by the continuous number line. Indeed, Kline (1959) compares the historical introduction of the irrational "number" to the introduction of "giraffe" into a concept of "animal" predicated on dogs and cows.

The following quote from Stifel (1544, cited in Kline, 1972) reveals the ongoing struggle with the concept of irrational numbers occurring more than fifteen hundred years after the Greeks had separated the contexts in which irrational and counting numbers arose. "Since in proving geometrical figures, when rational numbers fail us irrational numbers take their place and prove exactly those things which rational numbers could not prove ... we are moved and compelled to assert that they truly are numbers" (p.251). However, in attempting to represent irrationals as decimals, Stifel notes that "they flee away perpetually, so that not one of them can be apprehended precisely in itself" (p. 251), and therefore, he concludes that the irrationals are lacking in the precision requisite for status as numbers. Stifel expressed irrationals as decimals, but excluded them from the real numbers, asserting that the reals consisted of whole numbers and fractions. Here we see evidence of the difficulty involved in enlarging the concept of number. Like the giraffe in Kline's example which did not quite fit with a concept of animal based on dogs and cows, the irrational did not fit a concept of number based upon whole numbers and (by this time) fractions.

With irrationals (and with negative numbers as well) we are confronted with a kind of cognitive "ontogeny recapitulates phylogeny", as their inclusion as numbers requires reconceptualization of the conceptual schemas of our students in a matter of ten to twelve years commensurate with those historically requiring two millenia, all as a result of developing and reinforcing in the early school years a concept of number predicated upon cardinality, and thereby establishing a conceptual basis for the category of number too narrow to support its subsequent development. A conceptual framework predicated on cardinality is inadequate for the subsumption--even if correlative (Ausubel et al, 1978)--of fractions and irrationals, since these are derived from measurement rather than from counting. The result is a pedagogical dilemma of considerable proportions.

Davydov (1975) has resolved this dilemma by beginning in the first grade to stretch the concept of number beyond cardinality. His materials develop number from the activity of measurement rather than from counting. Figure 1 illustrates the task of measuring a line segment by a portion of itself arbitrarily designated as a unit. The result is a counting number "6".

```
|------|------|------|------|------|------|
 -unit-

          Measure = 6
```

Figure 1. Measurement of a Line Segment Resulting in a Counting Number

In Figure 2, the situation is complicated by the fact that after laying off the unit six times, there is a remainder of unknown measure "r" which must be compared to the unit in order to determine its fractional value. If the remainder can be laid off on the unit an integral number, say "n", of times (as in Figure 2), then the measure of the segment is 6 + 1/n. If the remainder cannot be laid off on the unit an integral number of times, then the process continues, with each new remainder being compared to the previous remainder functioning as the new "unit" until such a remainder is found. If the process continues to infinity and no such remainder exists, then the incommensurability of the original unit and the line segment is confirmed, and the measure of the segment is an irrational number.

Thus, counting numbers, fractions, and irrationals can be seen to arise naturally as the results of measurement.

```
|------|------|------|------|------|------|--
 -unit-                                   -r-

  r
|--
     r
   --
      r
     --
|------|
  unit
                 Measure = 6 1/3
```

Figure 2. Measurement of a Line Segment Resulting in a Fraction

Measurement then, rather than counting, provides an initial basis for the category of number sufficient to provide for its subsequent development without the necessity of successive reconceptualizations. This is a factor of enormous psychological consequence (See Skemp, 1987).

THE CASE OF MULTIPLICATIVE STRUCTURE

That this is the case became obvious in our research, conducted between 1988 and 1991, into the psychological structure of such a fundamental mathematical category as multiplication. The differences between American students' conceptualizations of this category, and those of Russian students using materials developed by Davydov and his colleague, L.K. Maksimov, were profound. For the Americans, who were secondary or university students, multiplication was invariably conceptualized as the repeated addition of (generally small) counting numbers. When the U.S. students were asked how they saw the algebraic formulation of the product of two real numbers "ab" as multiplication, those for whom the instance had any meaning at all substituted small whole numbers for "a" and "b", thereby subsuming the generalized product into their schema for multiplication predicated on cardinality. By way of contrast, the Russian students (from the fourth through the tenth form) conceptualized multiplication in its most abstract and generalized sense. Figure 3 shows a typical model drawn by children as early as the fourth form, which depicts "a" repeating "b" times.

```
|_a_|_a_| . . .|_a_|
.                 .
  .             .
    .         .
       .   .

      b times
```

Figure 3. Russian Pupils' Model of "a·b"

When the students were presented with an example of a binomial product, the results were similar. The U.S. students had been taught to find this product by multiplying the first, outer, inner, and last terms of the respective binomials (signified by the acronym, "FOIL"). Many had no idea how or why this process worked to produce the desired result. Those for whom the multiplication of binomials had any meaning at all substituted (mostly small) counting numbers for x and y. The most popular choices for x and y among university students were "2" and "1". The Russian students, however, conceptualized the binomial product as they did the product of monomials, in its most abstract and generalized sense. The model given by a fourth form child appears in Figure 4. The child first drew the scheme on the left and explained the required actions and the manner in which they produced the product. Then he substituted numbers for x and y to illustrate how this happened in a particular case (thereby demonstrating "the ascent from the abstract to the concrete" advocated by Davydov, 1972/1992).

```
(2·x + y)(x + 3·y) = (2·4 + 2)(4 + 3·2)
  \/   /  \   \/     \ /   /  \   \ /
   o  /    \   o      8   /    \   6
   \ /      \ /        \ /      \ /
    o        o          10       10
     \      /             \      /
       \  /                 \  /
        o                   100
```

Figure 4. Model of Binomial Multiplication by Russian Fourth Form Pupil

Every Russian child who had used Davydov or Maksimov's materials during their first three years of schooling was able to extend his/her knowledge validly and accurately into this new domain despite the fact that those in the fourth and fifth forms had not previously worked with binomial multiplication. The effect of using Davydov's materials during their first three years of schooling persisted for Russian ninth and tenth form students, despite the fact that they had experienced more or less traditional forms of mathematics instruction during the intervening years. The geometric representation appearing in Figure 5 is an example of the conceptualization of binomial multiplication demonstrated by these upper secondary students. It illustrates a strip of dimensions 2x + y by 1, repeating x + 3y times.

```
          x        3y                     x       3y
        ______________                  ______________
       |      |         |              ||     |         |
  2x   | 2x   |   6xy   |         2x   ||     |         |
       |      |         |              ||     |         |
       ---------------                 ||----|---------|
       |      |         |              ||     |         |
   y   | xy   |   3y^2  |          y   ||     |         |
       |______________|                ||____|________|
                                        1
```

Figure 5. Model of Binomial Multiplication by Russian Ninth Form Pupil

Thus, the importance of the initial development of categories is underscored, as initially instantiated schemas tend to be perpetuated. The Russian students' initial formation provided greater conceptual coherence, adequate to support the subsequent development of the category. That of the U.S. students did not. (For a further description of the abilities demonstrated by the Russian children, see Schmittau, 1993).

These studies of the psychological structure of multiplication were extended in our research to include exponentiation, which in the United States is taught as repeated multiplication, just as multiplication is taught as repeated addition. Thus, high school textbooks in the U.S. typically begin with examples such as

$$3^4 = 3 \times 3 \times 3 \times 3$$

and explain that the exponent denotes the number of times 3 is used as a factor. When we asked eight U.S. university students (two of whom were mathematics majors) for the meaning of "exponent", half of those who had majored in disciplines other than mathematics identified the meaning as "repeated multiplication" of positive integers. The others were unable to provide any meaning, however, designating the exponent merely as "a number which is written in the upper right hand corner next to another number". For these students a "cognitive entropy" of sorts was in evidence; whatever meaning multiplication as repeated addition of counting numbers had had for them was subsequently "damped out" with the extension to exponentiation as the repeated multiplication of counting numbers. In the end, the semantic content was reduced to nothing more than a syntactic cue.

The definition of exponentiation as repeated multiplication presents other problems, however. If 3^4

designates the repetition of three as a factor four times, what then is the meaning of 3^0, 3^{-2}, $3^{2/3}$, and $3^{\sqrt{2}}$? In order to establish some sort of meaning for non-positive and non-integral exponents, formal connections are typically presented in U.S. textbooks. It is asserted, for example, that

$$\frac{3^4}{3^2} = \frac{3 \times 3 \times 3 \times 3}{3 \times 3} = 3^{4-2} = 3^2,$$

so that the exponent of three resulting from the division is obtained by subtracting the number of times three is used as a factor in the denominator from the number of times three appears as a factor in the numerator. If the numerator and denominator are switched in the previous example, the result is

$$\frac{3^2}{3^4} = 3^{2-4} = 3^{-2} .$$

In this way, negative integer exponents are formally generated from positive integer exponents.

Zero as an exponent occurs when the numerator and denominator are the same power of the base. For example,

$$\frac{3^2}{3^2} = 3^{2-2} = 3^0 = 1.$$

Fractional exponents are <u>defined</u> so that $3^{a/b} = \sqrt[b]{3^a}$. Once the above exponents have been formally posited, students work many practice exercises with them. Typically, exponential functions are introduced later, often in a subsequent chapter, where students use the exponents developed formally (as above) to plot exponential functions such as $y = 3^x$, and then approximate irrational exponents from the resulting graph. Applied problems involving phenomena such as compound interest and population growth follow.

If the concepts involved in exponentiation and exponential functions are subjected to conceptual analysis, however, the notion of function emerges as an important underlying concept. If the historical development of the concept of function is then analyzed, it is found that it began to develop around the time of Galileo, out of attempts to express motion mathematically, and eventually to express the interrelationship of the variables involved in motion graphically. This effort to express motion mathematically provided a continuing impetus for the historical development of the concept.

The exponential function, of course, expresses the motion of growth, whether it be the growth of money when interest is compounded, the growth of bacteria or other populations, or similar natural growth situations. Unlike the historical development of the concept of number, which required successive reconceptualizations due to the inadequacy of its initial cardinal base, the historical development of the concept of exponentiation--although it began with positive integer exponents--proceeded through extensions and refinements. Thus, historical and conceptual analysis suggests beginning the development of the concept of exponent with the exponential function, and allowing the full range of real number exponents to emerge from the attempt to solve a problem in which it is required to express mathematically (both graphically and as a function) a situation of continuous growth. (It is important that the growth be continuous in order to provide for the emergence of non-integral exponents.) These considerations prompted the design of a teaching experiment which effectively reversed the order in which exponentiation and exponential functions are usually taught in U.S. classrooms. The design of this experiment appears within the larger context of the study described below.

A STUDY OF THE CONCEPTUALIZATION OF EXPONENTIATION

The study of exponential understanding was conducted with eight university students, who after responding to the question concerning the meaning of exponentiation mentioned above, were given the set of exponential exercises that appears below, for which they were asked to provide the answers.

$8^{2/3} =$

$\dfrac{2^2}{2^0} =$

$9^{-2} =$

$2^{\sqrt{2}} =$

$25^{3/2} =$

$2^{-6} =$

$6^2 \cdot 6^0 =$

$16^{5/4} =$

$(-4)^{-3} =$

$9^{2.5} =$

Table 1. Set of Exponential Exercises.

The students were also given a graph of $y = 2^x$ and asked to locate the points for which $f(x) = 2^0$, $f(x) = 2^{2/3}$, $f(x) = 2^{\sqrt{2}}$, and $f(x) = 2^{-3}$. In order to locate the last of these points, the student had to extrapolate the curve beyond the domain presented on the graph. Finally, the student was asked to solve two applied problems presented in Table 2. A clinical interview was conducted during which the student was asked to explain the solution of the tasks described above. This portion of the investigation was diagnostic.

Applied Problem #1. You accept a position in a lab and are assigned the job of checking bacteria cultures. On Monday you arrive and check the first culture. According to your best calculation it contains 2000 bacteria. You get busy and forget to check it on Tuesday. On Wednesday you estimate that there are now 4500 bacteria. Your boss wants to know how many were there on Tuesday, so you check with a co-worker and are told that bacteria populations increase exponentially with time. Is this information of any use to you in trying to answer your boss' question? In predicting how many bacteria will be in the culture on Friday?

Applied Problem #@. Mrs. Jones learns of an investment which will pay 10% on her money with guaranteed safety. Before investing she wants to know when her money will double. Can you help her?

Table 2. Applied Problems Involving Exponentiation

During a subsequent interview, the student was asked to complete the problem on plant growth presented in Figure 6. S/he was provided with a drawing of the growth pattern on successive days, and asked to answer the required questions. Obviously, the plant growth problem represented an idealized situation designed to promote exploration of the relevant mathematical structures, rather than to reflect botanical reality.

Plant Growth Problem Questions

At 8:00 Sunday morning a child notices a small plant growing near the steps of his house. He decides to measure it and finds that it is 3cm in height. He measures it again on Monday morning at 8:00 A.M. and finds it to be 9cm high. He decides to measure it at the same time on ensuing mornings.

Tuesday morning's measurement is 27cm and Wednesday's is 81cm.

Assuming that this growth pattern is descriptive of the entire growth history of the plant, how tall was it on the previous Saturday morning at eight o'clock? Why do you think he did not notice it?

How tall was it the previous Friday morning at 8:00? The previous Thursday at the same time?

If we want to show that Sunday was the first day the child measured the plant and be consistent with our numbering scheme, how should we number "Days" Saturday, Friday, and Thursday?

How tall was it at 8:00 the previous Saturday night? At 8:00 Sunday night? At 4:00 p.m. on Sunday?

When will the plant be 30 cm tall? 100cm tall? (Assume that it will grow to the size of a small tree).

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________________.________\|/________|_____|____|__


Day                       1         2     3    4

                         Sun        M     T    W
                         8AM       8AM   8AM  8AM
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Figure 6. Plant Growth Problem

This portion of the investigation was designed as a teaching experiment, with the interviewer working with the student in the zone of proximal development with respect to the concept, and providing only those questions necessary to create cognitive dissonance if the student failed to probe the full conceptual range of the problem.

After the completion of the "plant growth" problem tasks, the student was asked to "re-visit" the problems presented in Tables 1 and 2 and the graph of $y = 2^x$. The results obtained before and after working the "plant growth" problem are presented in Tables 3 and 4, respectively. In addition to their improved performance on the tasks chronicled in Tables 3 and 4, all subjects gave evidence of greater insight into the two applied problems as well.

EXPONENTS

	Number Correct	
	Before	After
$8^{2/3}$	2	8
$\frac{2^2}{2^0}$	5	8
9^{-2}	3	8
$2^{\sqrt{2}}$	1	4
$25^{3/2}$	3	8
2^{-6}	3	8
$6^2 \cdot 6^0$	5	8
$16^{5/4}$	2	8
$(-4)^{-3}$	2	7
$9^{2.5}$	2	7
	28	70

Table 3. Results for Exponential Exercises After Completing Plant Growth Problem

GRAPH OF EXPONENTIAL FUNCTION

Points	Number Correct Before	Number Correct After
$f(x) =$		
2^{0}	5	8
$2^{2/3}$	3	7
$2^{\sqrt{2}}$	2	6
2^{-3}	4	6
	---	---
	14	27

Table 4. Results for Exponential Graph Questions After Completing Plant Growth Problem

No explanation had been given of exponents, graphing, or applications during the teaching experiment. The student had simply worked through the questions asked in the "plant growth" problem. However, in so doing, s/he had been required to construct the meanings not only of positive integer exponents (meanings which, significantly, did <u>not</u> reflect "repeated multiplication"), but of negative integer, zero, and fractional exponents as well. Further, the student was required to develop the category holistically, so that the various types of exponents acquired their meaning not independently, and not through a linear formal generation from the repeated multiplication of counting numbers, but in their structural interrelatedness with all other constituents of the category. See Figure 7, where the results of the progressive answers to all but the final questions concerning the plant's growth are summarized.

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                                    |     |    |
________________.________\|/________|_____|____|_

Ht. 3^-2   3^-1  3^0  3^1/2  3^1  3^3/2  3^2   3^3   3^4
(cm)
     |     |    |    |    |    |    |     |    |
     |     |    |    |    |    |    |     |    |
                     |         |
Day -2    -1    0   1/2   1   3/2   2     3    4
                     |         |
                     |         |
Thurs    Fri   Sat       Sun       Mon   Tues  Wed
 8AM     8AM   8AM       8AM       8AM   8AM   8AM
                     |         |
                     |         |
                    Sat       Sun
                    8PM       8PM
```

Figure 7. Partial Results for Plant Growth Problem

The application of category theory to the genesis of mathematical categories reveals the primary cognitive difference between their formal and conceptual construction. The generation of number from counting, of multiplication from repeated addition of counting numbers, and of exponentiation from repeated multiplication, are all accomplished through the introduction of formal connections (often in the form of rules or definitions such as $\sqrt[b]{n^a} = n^{a/b}$). The result is a generative metonymic category structure which Lakoff (1987) reports is found only rarely in naturalistic settings. The holistic formation of such categories, however, reveals their <u>conceptual</u> essence and structural interrelatedness, thereby enabling their appropriation by students through what Ausubel termed "meaningful learning" (See Ausubel et al, 1978).

Before working the "plant growth" problem, students offered the following comments:

> S: They [exponents] tell you how many times you use [the exponent] as a factor...
> I: So $2^{2/3}$ tells you to use 2 as a factor 2/3 times?
> S: That's what I would think...
> I: How do you use something as a factor 2/3 times?
> S: Well, I would just take 2/3 of that number [an obvious confusion of exponentiation with simple multiplication].

Another student, a mathematics minor, said she was "not even sure what a function is", then described 2^3 as "2 times 2 times 2", but added, "I can't imagine something multiplied by itself 2/3 times." She continued:

> 2^3 is easy. And I remember the negative exponents mean the reciprocal. I feel comfortable with that to the same degree as I feel comfortable with 2^0, but it's

something I remember by rote memory. I can't say I truly understand the concept....2^x I can feel comfortable with because I think of x as being a whole number. But I'm uncomfortable with that because I realize the x could be something like 2/3. And even worse would be $2^{\sqrt{2}}$. I can't understand at all how you could find $\sqrt{2}$ of a number. To multiply it by itself $\sqrt{2}$ times just seems really weird.

One of the two mathematics majors in the group said initially that every type of number used as an exponent meant something different, an obvious reflection of the formal generation of exponents in school settings.

I: Are you saying that ... a positive integer as an exponent means one thing [and] if I have a fraction [as an exponent] it means another thing?
S: I guess that's the way it is, yes.

Virtually all the students felt that working the "plant growth" problem had enabled them to attain to a much different and more integrated understanding of exponentiation. In some cases the differences between the initial and final performance on the problem tasks was dramatic. One student who described himself as a "near perfect, 100% student in math" until calculus, missed nearly half the exponential exercises and location of points on the graph. He had very little insight into the two applied problems. After working through the "plant growth" problem, he got <u>all</u> of these problems correct.

The results were similar for two other students (non-mathematics majors like the student above), who missed virtually all of the exponential exercises and the location of points on the graph. One of them admitted to having developed a "total aversion to math after the eleventh grade" and consequently, taking no mathematics in college. The other described 2^0 as equaling zero and meaning "2 used as a factor zero times". Fractional, negative, and irrational exponents had no meaning at all. After completing the "plant growth" problem, both of these students corrected <u>all</u> of their mistakes. The second student was amazed that she had gone from having all the answers incorrect to getting them all correct. In working through the "plant growth" problem she commented, "I can't believe this tiny problem has gone through so many levels [of understanding]." At the end of the interview, the interviewer commented:

I: Now you notice after doing this problem with the plant, the little plant problem, you went back and corrected everything on these three sheets [i.e., on Tables 2,3, and 4].

S: I know. That's amazing. I want this to sink in. I'm sad that I didn't know this [before working the plant problem].

The two mathematics majors had virtually all the exponential exercises and graphing points located correctly. However, they still had some difficulties with the two applied problems. One had a great deal of insight into the source of the difficulty she experienced. She felt that her "meaning was rule-bound" and said she "tried hard <u>not</u> to worry about meaning because it can get in the way of efficiency". However, she had taken a graduate mathematics course in which some of the concepts inherent in the "plant growth" problem had been dealt with. She had also taught the topic as a teaching assistant in an undergraduate mathematics course. A portion of her initial interview follows.

I: What is an exponent?

S: Having taught that, I began with the idea of the historical development of what exponents were and that involved first the whole numbers. It began with the idea of natural numbers. They didn't even use the term "exponent". For instance, for 2^0 they would say "2 index zero"....Newton was much more explicit about using exponents in the way we think of using exponents -- fractional and negative-- ...[with] the idea of extending them from what was known and trying to make a consistent system.

I: What was known?

S: Even getting into the history of logarithms, we formed an association of an arithmetic progression with a geometric progression. The exponents formed the arithmetic progression and the number itself raised to the exponent formed the geometric progression. That's how they began to see how fractions satisfied the same consistency....As long as they formed a consistent pattern, you don't worry so much about what the meaning is.

I: So is that your understanding of what an exponent is, an arithmetic progression associated with a geometric progression?

S: Right.

I: So it wouldn't make any difference to you if you saw 2^3 or 2^0 or $2^{2/3}$ or 2^{-3} ? You would see them along the continuum of real numbers in the geometric sequence?

S: Right.

I: And you would see the exponents 3, 0, -3, and 2/3 as an arithmetic sequence also along the continuum of real numbers?

S: Right.

In spite of her understanding of the historical development of exponents as described above, she had not connected them with the problem of expressing a growth function. This the plant problem (by conceptually "telescoping" the historical development) required her to do. The problem was designed also to emphasize the arithmetic and geometric sequences which as this student correctly noted, functioned prominently in the historical construction of exponentiation. The numbers of the "days" in question constitute an arithmetic scale, while those of the "heights" constitute a geometric one. Nevertheless, when this student attempted to determine when the plant would be 30cm tall, she abandoned this "arithmetic sequence/geometric sequence" model and resorted to using logarithms. She observed a conceptual "gap" here, however, and said she noticed that such a gap occurred for her students also when they made the switch to logarithms to solve such problems in the classroom. What was interesting is that while she knew about the historical "arithmetic/geometric" sequence model, she did not actually use it! Since she "knew" that $3^{1/2}$ is $\sqrt{3}$, that $3^{3/2}$ is $\sqrt{3}^3$, and that the height of the plant at 4:00 p.m. Sunday would be $3^{1/3}$ which equaled $\sqrt[3]{3}$, she did not construct these meanings as the other students did.

The other students were required to notice that Day #1/2 (Saturday at 8:00 p.m.) occurred between Day #0 and Day #1 (Saturday and Sunday at 8:00 a.m., respectively), and to think about the fact that $3^0 \times k = 3^{1/2}$, while at the same time $3^{1/2} \times k = 3^1$. But $3^0 \times 3 = 3^1$. Hence, $k \times k = 3$ and k must, therefore, be $\sqrt{3}$. Now, since $3^0 \times k = 3^{1/2}$, and $3^0 = 1$ while $k = \sqrt{3}$, $3^0 \times k = 3^{1/2}$ can be expressed as $1 \times \sqrt{3} = 3^{1/2}$. Hence, $3^{1/2} = \sqrt{3}$. Similar reasoning was required for the construction of the meaning of $3^{3/2}$ as $\sqrt{3^3}$ and $3^{1/3}$ as $\sqrt[3]{3}$.

This student, as a mathematics major, simply remembered that $3^{1/2}$ was $\sqrt{3}$, for example, undoubtedly from the formal development of fractional exponents, and her conceptual "gap" became obvious to her when she approached the task of finding the time at which the plant would be 30cm tall. Thus, it is worth noting that during the historical development of exponentiation, particular attention was devoted to the arithmetic and geometric scales which appear on the x and y axes, respectively, but abstracted from the graphic representation of the function and juxtaposed in such a way as to facilitate the consideration of their relationship to one another. In working the "plant growth" problem the student is similarly required to juxtapose these scales (representing "day" and "height") below the drawing and to constantly refer to them as indicated in Figure 7. In addition, however, the student must continually refer these scales back to the problem in its original context, and thereby connect the exponentials s/he constructs with the exponential function itself and its meaning within the problem context.

It is important to note also that if the plant's heights are connected, the graph of $y = 3^x$ results. Thus, all essential components of the exponential function, its graphical representation, and the real-valued exponents themselves are seen in their structural interrelatedness within the context of a single problem reflecting the movement of continuous growth.

IMPLICATIONS

As Davydov (1972/1990) has pointed out, a genetic analysis encompasses both an epistemological and a psychological analysis. The former is concerned with the development of the concept as a sociocultural construct, while the latter focuses on the requisite cognitive actions required for the individual appropriation of the concept. This study, the results obtained from the analysis of U.S. and Russian students' conceptualizations of multiplication, and Davydov's (1975) work with the concept of number as well, suggest that it is not sufficient to simply replicate in the classroom the manner in which a concept developed historically (a practice to which U.S. educators are particularly prone on those rare relatively occasions when historical analysis is invoked at all). The historical development of the concept may not be worthy of replication; it may be flawed with respect to the coherence required for an adequate conceptualization of the category (as in the case of the historical development of the concept of real number).

It is for this reason that the historical analysis must be accompanied by a conceptual analysis (a research method which in the United States has unfortunately been largely relegated to the domain of philosophy).

Exponents provide an excellent illustration of abbreviated semiotic forms of thought which absorb into themselves the genesis of the concept they represent, thereby necessitating, as Davydov (1972/1988) has pointed out, a genetic analysis in order "to see in [these] abbreviated forms of thought its original course ... [and to] uncover the laws and rules of this abbreviating and then `recapitulate' the full structure of the processes of thought being analyzed" (p. 179). In the teaching experiment above, the student is required to recapitulate on an individual level the conceptual development of exponentiation reflected in its historical construction. The results of accomplishing the task set by this single problem corroborate Davydov's (1972/1990) contention that when the epistemological and psychological analyses are adequately accomplished, and the requisite pedagogical mediation is designed, the student may appropriate the concept by working one or at most a few problems. The pages of drill found so commonly in U.S. classrooms reflect the inadequacy of concept development in these classrooms.

The "plant growth" problem described above represents a task of considerable conceptual complexity. For this reason, although the results obtained after working it suggest that it was effective in accomplishing the task for which it was designed, it would seem desirable if such problems are used to introduce exponentiation in the classroom, to present students with perhaps one or two additional problems of similar conceptual complexity, in order to provide further opportunity for the mastery of the requisite conceptual connections.

Davydov suggests that through genetic analysis and adequate design of instruction, most students will attain to the requisite mathematical understandings. Indeed, in our study we found that even those students who began with little understanding advanced considerably in their conceptualization of exponentiation and exponential functions through working the plant growth problem.

Obviously, the "lengthy psychological research" (Davydov, 1972/1988, p. 196-197) required to restructure the mathematics curriculum along conceptual lines is not the "magic bullet" for which educators have been understandably seeking. But if through such research, mathematics may be made as accessible to the many as it presently is to the few, we can ill afford <u>not</u> to undertake it.

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HISTORICAL AND CONCEPTUAL ANALYSIS AS FOUNDATIONS FOR CURRICULUM: MULTIPLICATIVE STRUCTURE AS A CASE

Jean Schmittau

State University of New York at Binghamton

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