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Paper Title: A CONCEPTUAL CHANGE MODEL IMPLEMENTED WITH COLLEGE STUDENTS: DISTRIBUTIVE LAW MISCONCEPTIONS

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A CONCEPTUAL CHANGE MODEL IMPLEMENTED WITH COLLEGE STUDENTS: DISTRIBUTIVE LAW MISCONCEPTIONS

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In the past few years several categories of misconceptions as well as many specific misconceptions have been identified in the domain of mathematics (Confrey, 1990). One category of misconceptions involves overgeneralization of a rule or procedure, and one instance of this category involves the distributive law. The existence of distributive law misconceptions has been well documented (Matz, 1980; Markovits, Eylon, & Bruckheimer, 1983). This paper describes a teaching strategy that was used successfully with college precalculus students to amend distributive law misconceptions (Skane, 1993).

Background

The distributive property of multiplication over addition (Auffmann & Barker, 1987), also known as the distributive law, states that for any real numbers, a, b, and c, then:

$$a(b + c) = ab + ac;$$

$$(b + c)a = ba + ca.$$

The distributive law applies also to multiplication over subtraction, but in general does not apply to other functions or operations. Some students, however, mistakenly apply the distributive property to functions or operations that are not distributive. For example, Matz (1980) reported that some students wrote incorrect expressions of the form:

$$A(B \div C) = AB \div AC.$$

Markovits, Eylon, and Bruckheimer (1986) likewise reported that students made errors such as:

$\log(5) = \log(2) + \log(3)$,

$\sin(75) = \sin(30) + \sin(45)$,

$4^{2.5} = 4^2 + 4^{.5}$.

The authors have observed student mistakes such as:

$(x + y)^3 = x^3 + y^3$	(algebra)
$\log(x + 3) = \log(x) + \log(3)$	(logarithms)
$e^{x+2} = e^x + e^2$	(exponentials)
$2\sin(A + B) = \sin(2A) + \sin(2B)$.	(trigonometry)

Often such mistakes indicate an underlying misconception rather than just a thoughtless error. This misconception is compelling and acts as a barrier to learning the correct understanding of certain functions and correct ways to manipulate non-distributive functions or operations. This misconception impacts learning the logarithmic, exponential, and trigonometric functions; and this also leads to difficulty comprehending a fundamental expression used frequently in calculus, namely, f(a + h). Overgeneralization of the distributive property, like all true misconceptions, is resistant to change.

Sources of the Misconceptions

Several explanations have been offered for the source of distributive law misconceptions. Markovits, Eylon, and Bruckheimer (1983) postulated that misapplication of the distributive law to non-distributive operations or functions comes from a tendency to treat all functions as linear. Linearity has been explained by Matz (1980):

> Linearity describes a way of working with a decomposable object by treating each of its parts independently. An operator is employed linearly when the final result of applying it to an object is gotten by applying the operator to each subpart and then simply combining the partial results (p. 111).

For example, a student might erroneously compute the expression log(2 + 3) by first computing log(2), then computing log(3), and then adding the results.

Similarly, Movshovitz-Hadar, Zaslavsky, and Inbar (1987) suggested this misconception is an example of an overgeneralized principle, that is, applying something that works in one situation to another situation; they labeled such overgeneralized principles as distorted theorems or definitions. Likewise, Pines (1985) pointed out that students sometimes fail to understand that certain conceptual relations acquired in one framework may be inappropriate within another framework. Fischbein's (1987) notion of the primacy effect offers further insight into the resilience of this misconception. Sinch as a(b + c) and a(b - c) are forms encountered relatively early in mathematics education, a favored interpretation of a*(b Δ c) is (a*b) Δ (a*c), regardless of the meaning of "*" or "Δ".

Purpose

The focus of the present paper is to present the results of an investigation of the efficacy of a conceptual change model to amend distributive law misconceptions in a classroom setting with college precalculus students. The verb, amend, was carefully selected to mean change or modify for the better. The real hope was to achieve correction of the misconception for all students in the class and for all content of the precalculus course, but a more conservative approach was chosen due to the research (eg., Bell, 1983; Fischbein, 1987) that indicates that misconceptions are resistant to correction.

Method

A conceptual change model, described in detail below, was implemented three times in a classroom setting during a college precalculus course. Each time, after two weeks of study of a topic, students were asked to respond to a paper and pencil questionnaire that contained true-false statements, or computations, or mathematical expressions to be simplified. Also, student responses to items embedded in the precalculus final examination, and items embedded in a Calculus I pretest were analyzed for evidence of distributive law misconceptions.

Subjects

The subjects were all students (N = 68) enrolled in two intact sections of a college precalculus course at a large suburban community college located in a Middle Atlantic State. Initially there were 48 male and 20 female students, ranging in age from 18 to 50 years, with a median age of 21 years. All students had completed at least two years of high school algebra with grades of "C" or better. Some students had recently completed high school or college mathematics courses while others had not studied mathematics for as many as ten years.

Conceptual Change Model

The model of instruction selected to amend misconceptions about the distributive law was based upon the work of Rosalind Driver and her associates at the Center for Studies in Science and Mathematics Education at the University of Leeds in England.

Driver's conceptual change model, which Romberg and Tufte (1987) called a constructivist teaching sequence, consists of five stages of instruction: orientation, elicitation, restructuring, application, and review.

The teaching sequence, shown in Figure 1, was utilized after instruction on algebra, after instruction on logarithms and exponents, and after instruction on trigonometry.

<Figure 1 can be found at the end of this paper.>

Existence of the Misconception

Table 1 presents the responses of students to the first questionnaire, that was administered to them twice: once, prior to instruction, and again after two weeks of instruction.

Table 1

Algebra, Pre- and Post-Instruction: Percent of Distributive Law Errors (DLE) Per Item

		Instruction: Pre-	Post-
Item	True or False	% DLE	% DLE
1	$(x + y)^2 = x^2 + y^2$	19.1	2.9
2	$a(b - c) = ab - ac$	4.4	2.9
3	$\sqrt{x + 25} = \sqrt{x} + 5$	51.5	32.4
4	$x^3 + y^3 = (x + y)^3$	22.1	7.4
5	$f(a + h) = f(a) + f(h)$	95.6	86.8
6	$\|2x - 3\| = 2\|x\| + 3$	45.6	11.8
7	*If* $3 + \sqrt{3x + 1} = x$ *then* $9 + 3x + 1 = x^2$	44.1	38.2
8	$\sqrt{4x^2 + 25} = 2x + 5$	67.6	42.6
9	$x^2 + 9 = (x + 3)^2$	23.5	7.4
10	$A(X \div Y) = AX \div AY$	70.6	50.0

Note. N = 68. Item 2 is true; all other items are false. DLE = distributive law error.

Initially (pre-instruction) students responded to the questionnaire on the first day of the precalculus course, and the second time (post-instruction) students responded to the same questionnaire after a two week review of topics from algebra.

Throughout this paper a distributive law error is defined as an error consistent with misapplication of the distributive law to a non-distributive operation or function. Students committed other algebraic errors on the questionnaires, or in some cases did not respond to an item, but only DLE's are reported in this paper.

On the initial ten-item questionnaire only one student did not commit any distributive law errors; this was the only student who had participated in a pilot study for the project during the previous semester. Distributive law errors were committed by all others; 79.9% of the students committed 3 or more distributive law errors.

After two weeks of classroom instruction, the percent of distributive law errors for each item on the algebra questionnaire declined. Several items, however, still elicited distributive law errors from more than 30% of the students.

Another questionnaire was administered to the students after two weeks of instruction on logarithmic and exponential functions. As can be seen in Table 2, students committed distributive law errors for all except one item of the questionnaire. Only four items, however, elicited five or more students, (i.e., more than 10% of the students), to commit distributive law errors.

Table 2

Logarithms and Exponents, Percent of Distributive Law Errors (DLE) Per Item

Item	True or False	% DLE
1	$\log(A + 5) = \log(A) + \log(5)$	6.1
2	$\log \frac{A}{B} = \frac{\log A}{\log B}$	14.3
3	$3\log(A - 4) = \log(3A - 12)$	4.1
4	$\log(2A) = (\log 2) + (\log A)$	12.2
5	$\ln(A) + \ln(B) = \ln(AB)$	4.1
6	$\ln(x) - \ln(3) = \ln(x-3)$	2.0
7	$x^{3.2} * x^{0.8} = x^4$	6.1
8	$e^x * e^x = 2e^x$	10.2
9	$x^{2.5} = x^2 + x^{0.5}$	26.5
10	Compute: $\log(2.3 + 5.1)$	4.1
11	Solve: $\log(5x + 3) = 2$	2.0
12	Solve: $\log_2 x + \log_2(x + 2) = \log_2 3$	0.0

Note. N = 49. Items #4, 5, and 7 are true.

Individual student's distributive law errors on the questionnaire ranged from 0 to 8. The mean number of distributive law errors per student was 0.92 with standard deviation 1.52. After two weeks of instruction on logarithms and exponential functions 44.9% of the students made at least one distributive law error.

The one student who made the most errors on these items had been absent for six of the ten days of classroom instruction on logarithms and exponents. The other student who committed many (6) distributive law errors had also been absent several times during the two weeks of class instruction.

The results for questions involving trigonometry, (see Table 3) show that the mean number of distributive law errors per student was 1.13, with standard deviation 1.38 and 46.8% of the students did not commit any distributive law errors. On the other hand, more than half (53.2%) of the students made at least one distributive law error.

One student who made five distributive law errors actually used his own version of the laws of logarithms with distribution, instead of using the laws for trigonometric functions. During the small group discussions his classmates teased him about "mixing up stuff from a few weeks ago." The four students who made four distributive law errors had each been absent for at least one class during the instructional unit.

Table 3

Trigonometry, Percent of Distributive Law Errors (DLE) Per Item

Item	True or False	% DLE
1	$\sin(A + 10°) = \sin(A) + \sin(10°)$	6.4
2	$\sin(2A) - \sin(A) = \sin(A)[\sin(A) - 1]$	6.4
3	$3\sin(A + B) = \sin(3A + 3B)$	12.8
4	$\sin(A) - \sin(B) = \sin(A - B)$	14.9
5	$(1/2) \sin(2A + 30°) = \sin(A + 15°)$	10.6
6	$\sin 2A - \sin A = \sin A(2 - 1) = \sin A$	19.1
7	$\sin 2A + \sin 2A = \sin A(\sin A + 2)$	19.1
	Solve	
8	$\sin(A - \pi/4) = 0.5$	0.0
9	$1 - \sin(A) = \sqrt{3} \cos(A)$	23.4

Note. N = 47. Items #1 through #7 are false.

Intervention

Each of the first three questionnaires was utilized as a springboard (orientation and elicitation) for the constructivist teaching sequence. First in small groups and then in the whole class students discussed their responses (restructuring and application) to the questionnaires. They recorded in writing any changes in understanding (review) that occurred as a result of these discussions. As can be seen in the following tables, all students written comments indicated amendment of distributive law misconceptions immediately after the constructivist teaching sequence. Furthermore, the amendment persisted over time for most topics and for most students as documented by student's performance on unit tests given the following week, the precalculus final exam given at the end of the course, and the Calculus I pretest given seven weeks after the end of the precalculus course.

Squaring a Binomial

The topic, squaring a binomial, occurred in two forms during the study: first as a binomial expression in parentheses, such as $(x + y)^2$, and second, as squaring both sides of an equation. Students responded inconsistently to these expressions, therefore each form will be presented separately, beginning with the parenthetical expression.

Squaring a binomial in parentheses. Students' responses to squaring a binomial expression that is enclosed in parentheses are presented in Table 4.

Table 4

Percent of Distributive Law Errors (DLE) for Squaring a Binomial in Parentheses

Context	Task	Item	%DLE (N)
Pre-Inst.	T/F	$(x + y)^2 = x^2 + y^2$	19.1 (68)
Pre-Inst.	T/F	$x^2 + 9 = (x + 3)^2$	23.5 (68)
Post-Inst.	T/F	$(x + y)^2 = x^2 + y^2$	2.9 (68)
Post-Inst.	T/F	$x^2 + 9 = (x + 3)^2$	7.4 (68)
Post-CTS	T/F	$(x + y)^2 = x^2 + y^2$	0.0 (68)
Post-CTS	T/F	$x^2 + 9 = (x + 3)^2$	0.0 (68)
Exam	Solve	$x^2 + 6 = (x + 3)^2$	0.0 (45)
Calculus	Simplify	$(2a - b)^2$	0.0 (24)
Calculus	T/F	$x^2 + 16 = (x + 4)^2$	8.3 (24)

Note. CTS = constructivist teaching sequence; T/F = True or False; Inst. = Instruction.

Prior to any review or instruction, 19.1% of the precalculus students inappropriately distributed the square to each term of the binomial when the parenthetical expression was on the left side of the equality sign, and nearly a quarter (23.5%) of the students distributed the square when the parenthetical expression was on the right side of the equality sign.

After two weeks of instruction and review, several students still distributed the square to each term within the parentheses: 2.9% distributed when the parentheses was to the left of the equality sign, and 7.4% distributed when the parenthetical expression was to the right side of the equality sign.

In contrast, immediately after the constructivist teaching sequence, the percent of distributive law errors associated with a binomial expression in parentheses was reduced to zero. Furthermore, the amendment of the distributive law misconception for this topic was maintained over time. On the precalculus final exam, which took place thirteen weeks after the constructivist teaching sequence, the percent of distributive law errors for this topic was maintained at zero. Likewise, there were no distributive law errors on one item of the Calculus I pretest, twenty weeks after the constructivist teaching sequence, and only 8.3% of the students committed a distributive law error on the second item.

Squaring both sides of an equation. The procedure, squaring both sides of an equation that contains a binomial expression that is not enclosed in parentheses, is difficult for students. As can be seen in Table 5, prior to any instruction, 44.1% of the precalculus students squared each term of the equation separately. After two weeks of instruction, more than one-third (38.2%) of the students still operated on each term separately. Immediately after the constructivist teaching sequence, however, the percent of errors declined to zero.

The amendment of a distributive law misconception for squaring both sides of an equation was maintained over time by some, but not all, students. The data shows that 9.1% of the students committed a distributive law error when asked to solve

$$\sqrt{9x^2 - 4} + 1 = 3x$$

on the unit test, and 20.8% of the students erred on this same item on the Calculus I pretest. Half of the errors were due to removing the square root while squaring the other terms, and half of the errors were due to distributing the square root to each item under the radical.

Table 5

Percent of Distributive Law Errors (DLE) for Squaring Both Sides of an Equation

Context	Direction	Item	%DLE (N)
Pre-Inst.	T/F	*If* $3+\sqrt{3x+1}=x$ *then* $9+3x+1=x^2$	44.1 (68)
Post-Inst.	T/F	*If* $3+\sqrt{3x+1}=x$ *then* $9+3x+1=x^2$	38.2 (68)
Post-CTS	T/F	*If* $3+\sqrt{3x+1}=x$ *then* $9+3x+1=x^2$	0.0 (68)
Unit Test	Solve	$\sqrt{9x^2-4}+1=3x$	9.1 (59)
Post-Inst.	Solve	$1-\sin(A)=\sqrt{3}\cos(A)$	23.4 (47)
Post-CTS	Solve	$1-\sin(A)=\sqrt{3}\cos(A)$	0.0 (47)
Unit Test	Solve	$\sin(x)+1=\cos(x)$	0.0 (47)
Exam	Solve	$\sqrt{x+25}+1=7$	6.7 (45)
Exam	Solve	$\sqrt{2}\cos(A)=\sin(A)-1$	15.5 (45)
Calculus	Solve	$\sqrt{9x^2-4}+1=3x$	20.8 (24)

Note. CTS = constructivist teaching sequence; T/F = True or False; Inst. = Instruction.

Square Root of a Binomial

As can be seen in Table 6, simplifying an expression that involves the square root of a binomial elicited many distributive law errors. Prior to instruction, 51.5% and 67.6% respectively, of the precalculus students committed a distributive law error on items of the initial questionnaire that required taking the square root of a binomial expression under a radical. Even after two weeks of instruction, 32.4% and 42.6% of the students still committed distributive law errors on these items.

Immediately after the constructivist teaching sequence, the percent of error for taking a square root of a binomial was reduced to zero. Some students maintained an amendment of the misconception, but 20.8% of the students reverted back to their misconception as indicated by their performance on the Calculus I pretest. However, 40.9% of the calculus students who had not participated in a constructivist teaching sequence committed a distributive law error on this same item.

Table 6

Percent of Distributive Law Errors (DLE) for Taking a Square Root of a Binomial

Context	Direction	Item	%DLE (N)
Pre-Inst.	T/F	$\sqrt{x+25} = \sqrt{x} + 5$	51.5 (68)
Pre-Inst.	T/F	$\sqrt{4x^2+25} = 2x+5$	67.6 (68)
Post-Inst.	T/F	$\sqrt{x+25} = \sqrt{x} + 5$	32.4 (68)
Post-Inst.	T/F	$\sqrt{4x^2+25} = 2x+5$	42.6 (68)
Post-CTS	T/F	$\sqrt{x+25} = \sqrt{x} + 5$	0.0 (68)
Post-CTS	T/F	$\sqrt{4x^2+25} = 2x+5$	0.0 (69)
Unit Test	Solve	$\sqrt{9x^2-4} + 1 = 3x$	10.1 (59)
Unit Test	Prove	$1.4\sin(\theta) = \sqrt{0.04\cos^2(\theta) - 0.04 + 2\sin^2(\theta)}$	0.0 (47)
Exam	Solve	$\sqrt{x+25} + 1 = 7$	6.7 (45)
Calculus	Solve	$\sqrt{9x^2-4} + 1 = 3x$	20.8 (24)
Calculus	T/F	$\sqrt{x} + 2 = \sqrt{x+4}$	25.0 (24)

Note. CTS = constructivist teaching sequence; T/F = True or False; Inst. = Instruction.

Absolute Value

As can be seen in Table 7, nearly half (45.6%) of the precalculus students committed a distributive law error, prior to any instruction, when required to take the absolute value of a binomial expression. After two weeks of instruction, 11.8% of the students still erred on this type of problem.

After the constructivist teaching sequence, the percent of distributive law errors for taking an absolute value of a binomial expression was reduced to zero. The amendment of a distributive law misconception for this topic was maintained over time, as indicated by the performance of students on the precalculus final exam, and on the Calculus I pretest.

Table 7

Percent of Distributive Law Errors (DLE) for Taking the Absolute Value of a Binomial

Context	Direction	Item	%DLE (N)
Pre-Inst.	T/F	$\|2x-3\| = 2\|x\| + 3$	45.6 (68)
Post-Inst.	T/F	$\|2x-3\| = 2\|x\| + 3$	11.8 (68)
Post-CTS	T/F	$\|2x-3\| = 2\|x\| + 3$	0.0 (68)
Unit Test	Solve	$\|3x-2\| > 20$	6.8 (59)
Exam	Solve	$\|3x-2\| = 13$	0.0 (45)
Calculus	Solve	$\|2x-1\| = 3$	0.0 (24)

Note. CTS = constructivist teaching sequence; T/F = True or False; Inst. = Instruction.

Multiplication Over Division

As can be seen in Table 8, many (70.6%) precalculus students, prior to any instruction, committed a distributive law error when simplifying an expression involving multiplication over division. Even after two weeks of instruction, 50% of the students still committed a distributive law error for this topic.

Immediately after the constructivist teaching sequence, the percent of distributive law error for this topic was reduced to zero. Most students (all except 6.7%) maintained the amendment of the misconception for 14 weeks, as indicated by their performance on the precalculus final exam; 25% of the students, however, indicated the presence of the misconception by their performance on the Calculus I pretest.

Table 8

Percent of Distributive Law Errors (DLE) for Multiplication Over Division

Context	Direction	Item	%DLE (N)
Pre-Inst.	T/F	$A(X \div Y) = AX \div AY$	70.6 (68)
Post-Inst.	T/F	$A(X \div Y) = AX \div AY$	50.0 (68)
Post-CTS	T/F	$A(X \div Y) = AX \div AY$	0.0 (68)
Exam	Solve	$3(x \div 6) = 18$	6.7 (45)
Calculus	Simplify	$5(3x \div 10)$	25.0 (24)

Note. CTS = constructivist teaching sequence; T/F = True or False; Inst. = Instruction.

Student Evaluations

At the end of the Precalculus course students were asked to evaluate their experience of the constructivist teaching sequence. Every student (N = 45) enthusiastically recommended that this teaching strategy be used again.

Most students mentioned that the activity helped them understand and clarify details of the material, or find out their mistakes before it was too late, or it made them really think. Typical responses were:

> Henry: It was very helpful throughout the learning process, like a type of self analysis.

> Kate: It was a challenge coming up with a justification for a false answer, it was a good learning process, especially the discussions, I'm all for it.

> Doug: It lets the student examine their [<u>sic</u>] change in thinking

Several students mentioned that the activity was helpful because it was like teamwork that is necessary in the real world, and it allowed students to communicate with one another. For example, Steve wrote,

> Group discussion and group learning is rarely done in classes, but in the working world teamwork can lead to enhanced performance. It's called synergy, where the whole result is more than the simple sum of its parts.

John was even more emphatic:

> I haven't ever had an instructor do <u>anything</u> like that - ever. I'm not sure if it is because they didn't think of it or that they simply did not care enough. I think it helps in 3 areas:
>
> 1) Those who didn't know it & knew they didn't: they figure out <u>how</u> to do it.
>
> 2) Those who didn't know it & <u>thought</u> they did: enlightens them
>
> 3) Those who did know - It tightens the bow!

A few students mentioned the value of the group activity in terms of teaching and learning. For example, Frank said, "I can retain the information longer if I explain it to another student." And Will wrote:

> The discussions allow you to get an explanation or insight that may not come from the instructor. Instructors may sometimes take step [sic] and/or procedures for granted.

Similarly, Paula said, "Talking about problems is teaching and learning for students at the same time which offers better understanding. Info [sic] in layman's terms from fellow students is understood very easily."

Discussion and Implications

The distributive law misconception is wide spread among college precalculus students as evidenced by errors students committed on content from algebra, logarithms and exponents, and trigonometry. Traditional instruction is not a sufficient strategy to remediate distributive law errors for a substantial percent of students. A constructivist teaching sequence, however, was effective in amending misconceptions, was easy to implement in a college classroom setting, and had concomitant benefits such as revealing and correcting other mistakes.

The study was not designed to identify sources of distributive law misconceptions, however several factors emerged from the data that suggest avenues for further research. Students' performance seemed to be impacted by factors about the topic of study. For example, familiarity with the topic, familiarity with the order of presentation of the terms within a statement (eg., putting the more difficult expression to the left of an equality sign), and familiarity with the placement of parentheses within an expression increased student success on test items. Factors about the student, such as fatigue or rushing, also contributed to students' success or failure and hence to the reappearance of the misconceptions.

Recommendations

Students should be exposed to many forms and manifestations of mathematical expressions. For example, a set of problems in a textbook involving solving absolute value inequalities should vary the location of the absolute value expression on either side of the inequality. Throughout the study students described some expressions as "backwards" and therefore more difficult to recognize. For example, students learned and recognized that $\log(A) + \log(B) = \log(AB)$, but said that it was more difficult to recognize that $\log(AB) = \log(A) + \log(B)$. Students called the latter expression, "a backwards thing."

Instruction should provide students with more experience of the constraints and salient features of mathematical formulas. For example, students in the study were surprised that multiplication did not distribute over division; they were unaware of the constraints of the distributive law. Several students thought that the salient feature of a square root of a binomial was the plus or minus sign connecting the terms; these students did not realize that a radical is a grouping symbol.

Effective implementation of a constructivist teaching sequence requires allowing adequate time for students to discuss their responses, and requires adequate time for the review stage. When students omitted the review stage during the unit on trigonometry, the long term efficacy of the conceptual change was diminished.

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Figure 1

Conceptual Change Model	Implementation
Orientation: introduction to the content to be studied. The purpose of this stage is to create a sense of purpose and motivation for learning the topic of study.	The professor told students that a unit test would be given the following week and thus it would be advantageous to sharpen their understanding and to eliminate mistakes.
Elicitation: students make explicit their current ideas about the topic of study. The ideas are put out in the open and thus enable the instructor and others to detect misconceptions about the topic.	Students responded to a true/false questionnaire on content items and provided a written argument or reason for each response.
Restructuring: students clarify and exchange ideas with each other and with the instructor. The meanings students have constructed and the terminology they are using are compared with other, and possibly conflicting, meanings (Nussbaum & Novick, 1982; Rowell & Dawson, 1983; Stavy & Berkovitz, 1980). Some students will have had correct ideas from the start, while others will find that their ideas were inadequate or wrong. The instructor will promote conceptual conflict to help students recognize their misconceptions and thus those who realize their ideas were wrong will be receptive to change.	Students formed small groups of three or four people and discussed their responses to the questionnaire. Students negotiated arguments and answers until group agreement was achieved. Students then noted in writing any changes in thinking or understanding that occurred for each item as a result of the discussions.
Application: students use their developed conceptions in both familiar and novel ways. It is during this stage that the new or restructured conceptions of the topic of study are consolidated and reinforced.	After all small groups reached agreement on their responses, the professor led a whole class discussion about the items. Students presented as many different reasons as possible, and also observed similarities and differences among items on the questionnaire.
Review: students reflect on how their ideas have changed by comparing their thinking at this point with their thinking at the beginning of the instruction on the topic of study. The purposes of this stage are: 1) to make conscious the changes in thinking that have occurred; 2) to emphasize or reinforce any changes in conceptions that have occurred.	Students reflected upon the whole exercise and recorded any changes in their thinking. They responded in writing to the statements: I used to think ... Now I realize ... I have changed my thinking ...