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Some Misconceptions of High School Students about "Necessary and Sufficient Conditions"

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1. INTRODUCTION

Mathematics is a subject which is based on formal logic and hence uses formal symbols. This fact is often veiled by the emphasis generally placed in school, on the technical aspects of Mathematics. And no wonder, that generally, teachers refrain from introducing the logical aspects of Mathematics in class — it is very difficult for most students. We are satisfied if the students show — using SKEMP'S notions — (Skemp 5) an instrumental understanding rather than a relational understanding.

And yet, if we want our students to enter higher studies, with a sound foundation of Mathematics, we cannot in my opinion skip the formal logical aspects of Mathematics. The big problem is: how to do it? How to do it in a way which will not discourage students but on the contrary, will enhance their quest for deeper mathematical understanding. In my opinion, the solution is the method of `self-learning', and `learning by discovery', notions which were propagated by Lee Shulman and others during the sixties (Davis 2).

The main idea was to reduce to a minimum the frontal lessons and to let the students reach by themselves — with the help of a proper guidance — the topics which usually were taught during frontal `talk and chalk' teaching.

These ideas were of course contested by many educators, especially by Ausubel (Ausubel 1). The main objection was that, one cannot expect a student to work as a scientist — to behave constantly as if he were Archimedes shouting `Eureka' in the streets. Though it is an exaggeration, the opponents won. The `learning by discovery' method failed in general and quit the stage.

In my opinion the failure lied not in the idea of self-learning. The mistake was to think that it is possible to teach using mainly self-learning and `learning by discovery' methods.

These ideas look very appropriate to me. The problem lies in my opinion, in the right dosing. Most subjects of the standard curriculum are to be taught, more or less, in the conventional way. One cannot experiment too much on the students. But certain subjects, especially the non-standard ones, can and should be taught through self-learning tasks.

This way of learning changes the atmosphere in the class. It is a blessed deviation from the routine. It challenges the intellect of students, and in my experience students are thrilled with such challenges. It looks as if they are receiving credit from the teacher, that they are able to learn by themselves and do not have to rely merely on him. They show more readiness to cope with difficult problems. And, therefore, this in my opinion, is the proper way to teach the formal logical concepts of Mathematics.

For my first experiment, I chose the concepts of `necessary' and `sufficient' conditions.

Many people, not only in the field of mathematics, confuse these concepts. Very often we encounter the tendency to regard every condition automatically as both a necessary and sufficient condition. The problematics which appear in dealing with statements of the form `If A then B', were already discussed by O'Brien (O'Brien 3). He discerned difficulties, not only with adolescents, but with college students as well.

People, in general, tend to simplify matters, and this simplification causes confusion. In, non-mathematical every-day speech, we usually phrase statements as a sufficient condition, meaning both sufficiency and necessity. For example, when a father tells his son: "If you'll wash the car you'll get $5." He means, of course, that if his son won't wash the car, he'll get nothing. But in formal logical sentences when we write: `If A then B', this does not imply that `if not A then not B'.

In dealing with mathematical statements the distinction between necessary and sufficient conditions is crucial, and the teacher must not skip pointing out this distinction.

It is easier for the student if a mathematical condition is both necessary and sufficient, it then saves too much thinking. How easy it would have been for the student if the vanishing of the derivative at a point were a necessary and sufficient condition for an extremum point. Therefore, students often fail to discern a stationary point which is not an extremum point.

The aim of my experiment was to test how students scope with self-learning of formal symbolic concepts, concepts the definitions of which are presented in a rigorous symbolic form. Symbols (letters in this case), presented general abstract statements, the content of which is irrelevant.

This naturally presents a problem for students and poses a new situation for them. Till now, they were used to regard letters as standing for numbers, points, functions, but not abstract statements. But these symbolic forms are the essence of mathematical logic. Therefore, the main target of my experiment was to learn how students cope with the above mentioned difficulties through self learning. Interesting misconceptions turned up during this experiment — ones that may not turn up during frontal teaching because teachers often just automatically give students the right answers. We do not give the students enough time to digest the new material themselves, and therefore cannot bring out and focus on the intrinsic misconceptions which are bred in the students' minds during their efforts to understand new concepts.

2 METHOD

The questionnaire was given to 44 10th grade students at the Israeli Academy for Sciences and Arts. These students were not the students who excel in mathematics, although most of them intend to matriculate in the five-points mathematics test, which is the highest level of mathematical matriculation tests in Israel. The majority of these students excel either in the fields of music, the plastic arts, chemistry, or in biology — scientific subjects which do not require too much mathematics (as for instance in physics).

In this school prevails an atmosphere which encourages non-routine ways of studying, and hence produced a positive attitude toward the questionnaire (more details will be discussed later). I mention this attitude, because any difficulties which will arise during the task will point at real difficulties and will not be attributed to a frivolous attitude of the pupils.

The questionnaire was built in such a way, so that at the beginning appear the formal definitions, and the students are immediately asked to give appropriate examples. In this way I can check if the definitions were understood, at least on a basic level. The questionnaire continues with more complicated questions which examine the deeper understanding of the definitions.

The first five questions do not deal with mathematics. Only the sixth question deals with a mathematical instance of a necessary and sufficient condition.

I was not particular about the correctness of non-mathematical examples. If, for instance, a student wrote that if the sun shines, this is a sufficient condition that it is still day, I took it as a correct example, not taking into account the rare phenomenon of the Scandinavian midnight-sun. Or if a student wrote that the boiling of water is a sufficient condition to the fact that the temperature of the water reaches 100° Centigrade, I didn't mind that the student disregarded the effect of pressure. The emphasis lied on the logical understanding.

The answers were divided into 3 categories. In the first category I included the fully correct answers. In the second category I included the partly correct answers, meaning answers which showed a basic understanding but with weak reasoning or no reasoning at all. This kind of answers was common when the answer seemed (intuitively) very obvious. In the third category I included incorrect answers or when there was no answer at all. Answers of the form `yes' or `no', even if they were correct, were put in the third category when the reasoning was totally wrong. (Maybe the `yes' or `no' answer was derived from pure guessing).

A table showing the distribution of the answers among the categories appears at the end of the questionnaire.

3. THE QUESTIONNAIRE

I'll now present the questionnaire, including the role of any exercise and tasks analysis, (which, of course, were not included in the original questionnaire). The remarks about the task analysis will be parenthesized.

A,B will represent in this questionnaire attributes, situations or events. We'll call then by the mathematical term: `conditions'.

First Definition: A is necessary for condition B, if B cannot exist without the existence of A. (There arises some difficulty in grasping a definition which includes two negatives (cannot....without). I deliberately chose this definition in order to create a difference between the definition of a necessary condition and that of a sufficient condition. This was done in order to test whether the students could connect between these two different definitions).

An Example: A – the sky is cloudy; B – it is raining.

Remark: A can exist without B. Perhaps the sky is cloudy, but it doesn't rain. So even if A is necessary for B, one cannot derive the existence of B from the existence of A.

(A hint toward the difference between being a necessary condition to being a sufficient one).

Exercise 1: Give two examples of pairs A and B, so that A is necessary for B (I expected that in at least one example A will be only necessary for B).

Exercise 2:

a) A is necessary for B. A does not exist. Is it possible that B exists? Justify your answer. b) A is necessary for B. B does not exist. Is it possible that A exists? (The student should show understanding of the different roles of A and of B).

Second Definition: Condition A is sufficient for condition B, if from the existence of A, one can conclude the existence of B. Compare with the first definition.

(I deliberately chose here A for the sufficient condition — though A stood in the first definition

for the necessary condition — in order to test if students understand the irrelevance of the form of the symbols in a formal definition. We'll see later on if this caused confusion).

An Example: A – Ron is a mathematics teacher. B – Ron knows the Pythagorean theorem.

Remark: If A is sufficient for B, B can exist even without the existence of A. Maybe a person knows the Pythagorean theorem, without being a mathematics teacher. This is precisely the difference between a necessary and a sufficient condition.

Exercise 3: In the first example (A – cloudy sky, B – it is raining), which condition is sufficient to which? (testing if the students understand the difference and hinting toward the fact that if A is necessary for B, then B is sufficient for A).

Exercise 4: Give two examples of pairs A and B, so that A is sufficient for B.

Exercise 5:

a) Is A necessary and sufficient for itself?

b) Give two examples of pairs of conditions, A and B, so that A is both necessary and sufficient for B. Try to give a non-mathematical example. For a mathematical example consider the congruence theorems. (Do students understand that to be a sufficient condition is not the opposite of being a necessary condition).

Exercise 6: This time we will deal with a mathematical example. Let A stand for: `N is an even number,' and let B stand for: `N is a multiple of 4.'

a) Is A necessary for B?

b) Is A sufficient for B?

c) Is B sufficient for A? Give a mathematical reason for your answers.

(Can students perceive the difference between the two kinds of conditions when they deal with purely mathematical concepts? The main goal is to make them understand this difference when working with mathematical concepts. In (b) I expect the students to give a counter example of an even number which is not a multiple of 4).

Exercise 7: Which of the following statements is true for any conditions A & B?

a) If A is sufficient for B, and B is sufficient for A, then A is both necessary and sufficient for B.

b) If A is both necessary and sufficient for B, then so is B for A (Testing if the students grasp the symmetry of this relation).

c) In order to show that A is both necessary and sufficient for B it suffices to show that A is sufficient for B, and B is sufficient for A.

(To show sufficiency according to the previous definitions is easier than to show necessity; this exercise leads to the fact that showing necessity directly is actually superfluous).

Exercise 8: (Taken from O'brien (Obrien 4)

Four cards are presented to you. Each card has a number written on one side and a letter on the other side. A sufficient condition for an even number to be written is that the letter will be taken from the letters A-I. Which of the cards must be turned in order to either affirm or contradict the above statement?

G	18	7	T

(A non-routine instance of a sufficient condition, as it is a kind of a mathematical amusement question, it was chosen to appear as the last exercise when the students are already exhausted. I expect the students to point out cards G and 7, and to explain why turning 18 and T is irrelevant.)

4. RESULTS AND ANALYSIS

The following table shows the distribution of the answers according to the 3 categories mentioned before.

N = 44

Exercise No.	Number of Answers in Category I	Number of Answers in Category II	Numbers of Answers in Category III
1	27	17	0
2	10	30	4
3	36	0	8
4	30	7	7
5 (a)	13	21	10
5 (b)	24	7	13
6	4	27	13
7 (a)	11	18	15
7 (b)	7	11	26
7 (c)	8	14	22
8	22	10	12

I will now analyze the answers to each exercise separately.

Exercise 1: All the examples were actually correct though some of them I classified under the second category, as they seemed to me artificial. There were some nice examples as: A — I exist, B — I think, or: A — there is a fire, B — there is smoke (following the famous saying: "Where there's is smoke, there's fire"). Some students gave examples where the conditions were both necessary and sufficient, as: A — the temperature of the water reached 100° Centigrade, B — the water started to boil. There were also examples which were analytically true (according to Kant) meaning that the necessity of the condition was derived from the semantic meaning of the words which describe the condition. As for example: A — the man is bald, B — the man has no hair. Instances for artificial examples which I classified under category II are: A — the boy has a balloon, B — the boy inflates a balloon. Or: A — the child went into the garden, B — the child is in the garden.

The majority of the examples were non-mathematical, but there were also mathematical examples as for instance: A — the quadrilateral has equal sides, B — the quadrilateral is a square. No doubt that the definition of a necessary condition was understood, at least on a basic level.

<u>Exercise 2</u>: In order to answer correctly, the student has to show some ability of dealing with mathematical formalities.

a) 40 (out of 44) gave a fully correct answer, namely, that from the definition of a necessary condition one can conclude that if A does not exist then B doesn't exist. (This is actually a repetition of the definition). Four students wrote that B can exist even without the existence of A, contradicting the definition of the concept. These students' mother tongue was not Hebrew, and as in formal definitions the exact semantic meaning of the words is very crucial, one can understand the difficulty which confronts students who do not master the language in which the definitions are given.

b. Only 10 students gave a fully correct answer. They actually used my remark following the definition. Thirty students just wrote the simple answer, "B must not exist".

Here we meet a common phenomenon. When students are asked about a statement which sometimes is true and sometimes not, they do not know that they have to give an example for each case. They regard an answer of the form, "It may be true", as a fully justified answer.

Exercise 3: An easy exercise. 36 students answered correctly. As I mentioned, when presenting the questionnaire, I deliberately marked the sufficient condition in the second definition with the same letter (A), as I marked the necessary condition in the first definition. The 8 pupils, who gave the wrong answer, were influenced by the letter standing for the sufficient condition in the definition, disregarding the fact that in this instance A stands for the necessary condition.

Among the right answers there were two students who interchanged the letters (A stood for `It is raining', and B stood for `The sky is cloudy'), and so dispelled any confusion which may arise. In my opinion this is an indication of correct understanding. In contrast to them, the 8 students who gave the wrong answer showed a poor learning process according to Ausubel (Ausubel 1). Instead of paying attention to the real meaning of necessary and sufficient conditions, they gave their attention to the letter which stands for the condition in the definition. They showed in this case no ability to cope with formal definitions.

Exercise 4: As the role of Exercise 3 was to hint towards the fact that if A is necessary for B, the B is sufficient for A. I expected the students to interchange the roles in their examples in Exercise 1. Nine students answered in this way. All in all 30 students gave correct examples. Seven students gave only one example instead of two as requested, or at least one example was artificial (A — the boy threw the ball into the basket; B — the boy played basketball). I classified these answers under category II.

Seven students confused the two kinds of conditions or gave wrong examples. For instance: a sufficient condition enabling a man to lift heavy weights is that he is strong. (This is a necessary condition — not every strong man lifts heavy weights). There were also mathematical examples which were wrong. Having equal angles in a polygon is sufficient for the polygon to have equal sides (This is true only in a triangle). These answers were classified under Category III.

Exercise 5: a) This exercise needs some ability to work with a formal language. Only 13 students answered that A cannot exist without the existence of A, and if A exists it is clear that A exists. This kind of answer I classified under the first category. Twenty-one answers I classified under category II. Actually, it is more appropriate in this case to subdivide category II into two sub-categories. One category for those who gave an example instead of a general proof (four pupils) and the other category for those who gave no reason at all. A typical example was: If Ron is a mathematics teacher, he clearly is a mathematics teacher. Actually, this example exhausts everything. Nevertheless, I wanted a general answer using a general symbol for the condition.

The second sub-category (17 students) which constitutes the majority of category II is comprised of those students who gave no reason for their (correct) answer. It seems to me that when one encounters a tautology, it seems so obvious that no justification is needed. All in all, I conclude that the majority of students, comprising the first two categories (34), understand that A is both necessary and sufficient for itself.

Under the third category I classified the students who gave no answer at all, or gave a wrong answer. Among the wrong answers were: "A maybe necessary and sufficient for itself' and `The second A does not necessarily stand for the same condition as the first A'. Such an answer shows that the student doesn't know how to handle formal statements. He doesn't know that in the same context, the same letter stands for the same condition. Here we have one more example of students who are confused when dealing with a strict formal language, like in mathematical logic.

b) Twenty-four students gave correct examples, mostly from the field of mathematics. They followed my instruction and explained correctly why the conditions in the geometrical congruence theorems are both necessary and sufficient. There were other examples as well. For instance: A — the triangle has equal sides; B — the triangle has equal angles.

Most of the non-mathematical examples were actually analytically true (see my remarks for Example 1). These kinds of examples were classified under Category II. It is very difficult to find a non-mathematical example for both necessary and sufficient conditions which are not of the above mentioned form. A good example, not from this form, is: A — the man is alive; B — the man breathes (Given by one of the pupils). In a frontal lesson I would have mentioned the fact, that a condition which defines a concept is automatically both necessary and sufficient.

Under category III were classified students who didn't answer at all or gave wrong examples. For instance: A — Joe is studying at a particular university; B — Joe got high grades in his matriculation exams. B is necessary for A but not sufficient. Not everyone who succeeded in the matriculation exams studies at this particular university.

<u>Exercise 6</u>: Here I test how students handle necessary and sufficient conditions of pure mathematical concepts. Thirty-one students answered correctly, but the majority of them (27) remarked in part (b) that an even number doesn't have to be a multiple of 4, and didn't give any example. Again, it seemed so obvious to them, that they didn't see any reason to justify their answer with a proper example (See my remarks to Exercises 2 and 5). I classified these answers under category II.

Under the third category I classified no answer at all or wrong answers. As students usually answered in the affirmative to (a), some of them answered in the affirmative also to (b). Here, I encountered an interesting wrong argument.

As in non-formal speaking, the term `necessary' has a stronger meaning than the term `sufficient', some students argued that if a condition is necessary it must also be sufficient — which contradicts by this argument, previous examples to the contrary (more about it in the following discussion).

Only one of the students answered incorrectly because she didn't quite understand the exact meaning of the term `a multiple of 4'.

<u>Exercise 7</u>: This is the hardest nut for the students, and no wonder, as this exercise sums up all the general formal connections between the two kinds of conditions. Only 6 students answered correctly all parts of the exercise, and were classified under category I. a) Eleven students were classified under category I for a full (including argument) answer. They answered by returning to the definitions and showing that if A is sufficient for B, then B is necessary for A. Eighteen students wrote that the statement in (a) is true without any argument and were classified under category II.

The remaining 15 students gave no answer at all or argued that the statement was false. They didn't give any reason and were of course classified under category III. I classified under this category answers which gave particular examples and avoided the restriction in the exercise to prove generally b) The goal was here to lead students who answered correctly to part (a) to the fact that equivalence of conditions (both necessary and sufficient) is a symmetric relation.

Here I found an inconsistency in the answers of those students who answered to part (a) correctly but without any argument. I conclude that these students guessed the answer to part (a) without understanding why, as the answer to part (b) is directly implied from (a) (for those who understood why (a) is true.)

Only 7 students answered correctly including an argument which was similar to that of part (a). (Yet, 4 students who gave a full answer to part (a), did not see the connection). Eleven students wrote that the statement is true without any argument.

The remaining twenty-six students gave wrong answers or no answer at all. A typical wrong argument was, that if A is both necessary and sufficient for B, then B must be either necessary or sufficient for A, but not necessarily both.

One student who had some difficulty coping with both kinds of conditions simultaneously divided the question into two parts — and hence her answer, which was correct relative to this division. She divided the question in this way. If A is necessary for B, does it imply that B is necessary for A? And the same for sufficiency. And of course, according to previous examples the answer was in the negative, which is correct relative to this phrasing of the exercise.

Another kind of wrong answer was that a condition cannot be both necessary and sufficient, despite all previous examples (Further discussion will follow in the discussion section).

c) Although this part follows immediately from part (a) only 8 students noticed it. Fourteen students answered in the affirmative without any argument. The other half answered incorrectly, usually as in part (b).

Exercise 8: The students liked this exercise, and found an element of amusement in it. Twenty-two (exactly half), answered correctly that one has to turn over cards G and 7. Most of them also argued correctly, namely, that if there is an odd number on the other side of G, the statement is contradicted, and if the letter on the opposite side of 7 is among the letters A-I, the statement is also contradicted. Except for one or two students, no one explained why it is irrelevant to turn over the other two cards. Nevertheless, I classified these answers under the first category.

Five students answered that the only card which has to be turned over is card G, and gave the right reason. Five students claimed that one must turn over cards G and 18, arguing that if the letter on the other side of 18 is among A-I, we have verified the statement. One of these students added an argument why we do not have to turn over cards 7 and T. `Seven is not an even number and T is not among the letters A-I. I classified all these questions under category II.

In this category no one wrote that card 7 has to be turned over. Maybe that's because the statement mentioned only even numbers, the students thought that an odd number is irrelevant.

Another explanation for not perceiving the relevancy of card 7 is the difficult of using negation. `A implies B' is logically equivalent to `not B implies not A'. If A is the statement: "The letter is among A-I" and B is the statement: "The number is even", then the logical implication of the claim in the exercise is that if the number is odd then the letter on the other side must be among J-Z. This is actually how one has to argue why we must turn up card 7. This proved to be difficult for some students. Under the third category I classified 5 students who didn't answer at all, and 7 students who wrote that one has to turn up cards 18 and T. The students' arguments pointed to the fact that they confused sufficiency with necessity. If the claim was that a necessary condition to an even number is a letter among A-I they would have been right. In spite of all the previous examples and exercises, this confusion still remained.

5 DISCUSSION

From the table which shows the distribution of answers to each question among the 3 categories, we see that the main difficulties arose in exercises 2,5,7 and to some extent also in exercises 6 and 8. And no wonder, since these are the exercises where the student has to show ability in handling a formal language. The basic level of understanding was reached by most students as is shown from the table by the distribution into categories of the answers to the easier exercises.

Only about a quarter of the students gave a full correct answer to at least two of the exercises 2,5,7. Exercises 6 and 8 do not deal exactly with general formal symbolic language, but request a deeper insight into the meaning of necessary and sufficient condition.

From the students' answers we can learn something about the levels of ability of pupils, which resemble in some way the levels of cognitive development according to Piaget (Sutherland 6). Most students deal with these kind of exercises in a way which resemble the `concrete' operation of Piaget. They think and reason through concrete examples. When they have difficulties in proving a general symbolic statement they bring examples of this general statement which prove (in their eyes) the statement. Their examples are correct and hence show that the students understand the new concepts. The difficulty lies in their ability to work with a symbolic formal language. The few students who showed the desired ability think, in some way, on a level which resembles the `formal' activity of Piaget.

A good touchstone for this point was when students were confused by the fact that the same letter stood for different kinds of conditions in the two definitions.

Category II, which includes most answers, is actually the category of students who understood the new concepts but had difficulties in handling them in a formal symbolic language. According to Skemp's terminology (Skemp 5) they showed `instrumental' understanding, and not `relational' understanding. They showed that they know how to work with the new concepts (see the answers to exercises 6 and 8), but had difficulties to formally explain why their arguments were true.

Another difficulty encountered was in handling simultaneously two concepts and making the connection between them (this was needed in exercise 7). This resembles the `coordination' problem of Piaget; in this instance an abstract coordination in the pupil's mind. The students had immense difficulties dealing simultaneously with the two statements "A is both necessary and sufficient for B" and "B is both necessary and sufficient for A". In order to argue correctly in this exercise one has to "toss" the definitions from side to side in his mind.

Some students showed also a tendency to simplify matters in order to avoid difficulties, especially on the coordination problem. For instance, deciding that a condition is either necessary or sufficient, but not both, or dividing the problem in exercise 7(b) into sub-problems (see the analysis of Exercise 7b).

The everyday way of speaking which is not strict about formal correctness does not help either. As the word `necessary' has a stronger connotation than the word `sufficient', some students concluded that a necessary condition must also be sufficient.

I want to conclude this discussion about the method of self-learning. In this method students are confronted with certain difficulties, but, in my opinion, these are difficulties which students have to be trained to cope with. As such a self-learning task is not an examination, I see no harm if it appears difficult and I don't grade the pupil. Through such tasks we perceive difficulties of students which escape our attention during frontal teaching, or we perceive them only through the examinations, and then it may be too late.

From the students' attitude towards this task (and other self-learning tasks with which I worked with pupils), I learned that it presents to them a welcome challenge. It seems to them as if they get approval from the teacher on their learning abilities. They were thrilled with these kinds of tasks and asked for more.

During frontal teaching teachers are sometimes carried away and do not allow students the needed time to cope by themselves with new material. Without noticing we tell the students (or at least hint) the correct answer. So the student gets the impression that he shouldn't give too much effort. And the results are usually that the learned subject is not properly rooted in the pupil's mind. But, if the student has to work hard by himself when encountering the basic definitions of new concepts, they may become rooted properly in his mind, and thus we may avoid misconceptions in the future. In mathematics' lessons which took place after this task I noticed more awareness towards the concepts of necessary and sufficient conditions.

Nevertheless, I do not delude myself that after performing this task, students will master these concepts. To apply these formal concepts correctly at every instance of mathematical condition it needs more time and experience. Yet, this is, in my opinion, a right step in the right direction.

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